9 Strength Theories of Lamina

STRENGTH OF ORTHOTROPIC LAMINA

For isotropic materials the simplest method to predict failure is to compare the applied stresses to the strengths or some other allowable stresses. In this case there is no principal material direction so the material strengths are the same in all directions. For isotropic metals failure usually occurs by yielding and can be simply predicted by the *maximum shear stress theory*. For a plate in plane stress conditions the principal stresses σ_{I} and σ_{II} can be found half their difference can be compared to the yield strength of the metal. The criterion for failure is

$$\frac{\sigma_{I} - \sigma_{II}}{2} \geq \left|\tau_{yield}\right|$$

or

$$\sigma_{I} - \sigma_{II} \geq \left|\sigma_{yield}\right|$$

which is represented graphically in Fig. 9-1.

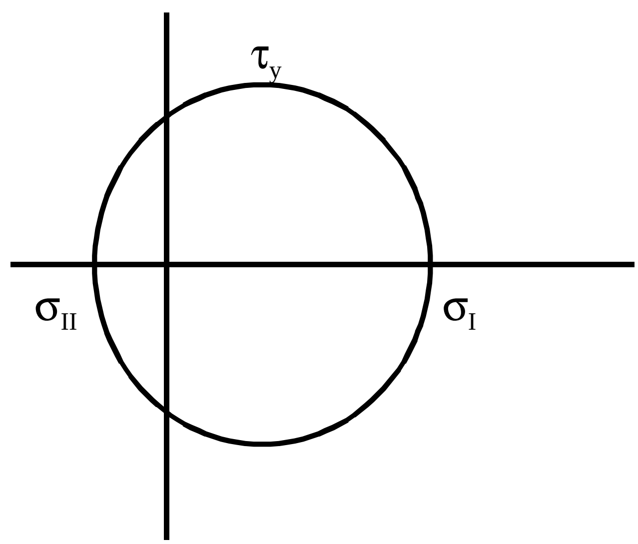

Figure 9-1. Yield criteria for biaxial stress

A better criterion for most ductile metals is the *distortion energy theory* that states failure occurs when

$$\sigma_{I}^{2} + \sigma_{II}^{2} - \sigma_{I}\sigma_{II} \geq \sigma_{yield}$$

For orthotropic composite lamina these methods are not sufficient because the failure mechanisms and strength properties change with direction of loading. Failure usually does not occur by yielding but rather by fracture of one of the constituents or the fiber-matrix interface. Unlike isotropic materials, and axis of maximum stress does not necessarily coincide with direction of maximum strain. As a consequence the highest stress on body may not be the highest critical stress in the structure. The notation often used to describe strength properties of orthotropic lamina is shown in Fig.9-2. The ultimate tensile strength in the fiber direction is denoted by *X,* the ultimate longitudinal

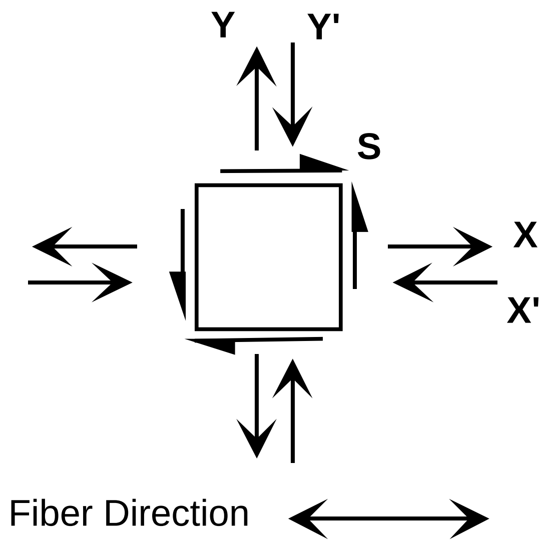

Figure 9-2. Strength notation for orthotropic lamina

compressive strength by *X'* , the transverse tensile strength by *Y* , the transverse compressive strength by *Y'*, and the shear strength by *S*. In orthotropic composites the tensile and compressive strengths in the principal material directions usually do not have the same values. This is because the mechanism of failure can change from fiber fracture when in tension to fiber microbuckling and interfacial splitting when in compression. However, positive and negative shear strength in the principal material direction is the same. This can be seen in Fig 9-3, where the fibers and matrix experience the same load

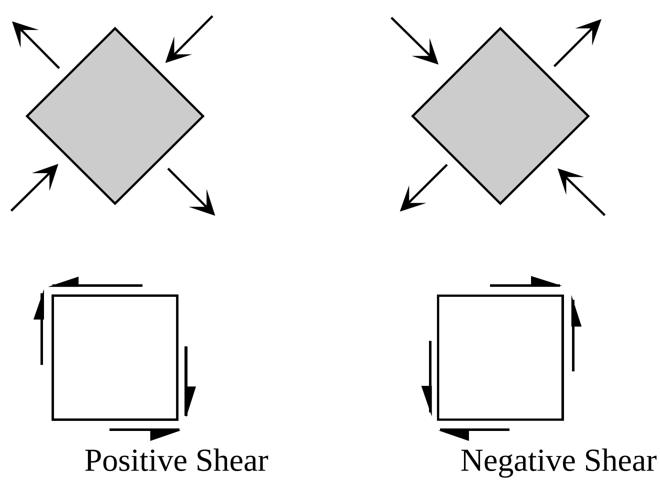

Figure 9-3. Comparison of positive and negative shear for stress in the principal material directions.

orientation regardless of the sign of the shear stress.

The shear strength for off-axis shear, say at 45º to the principal material direction, can be different. This case is illustrated in Fig. 9-4. In this case positive shear places the fibers in

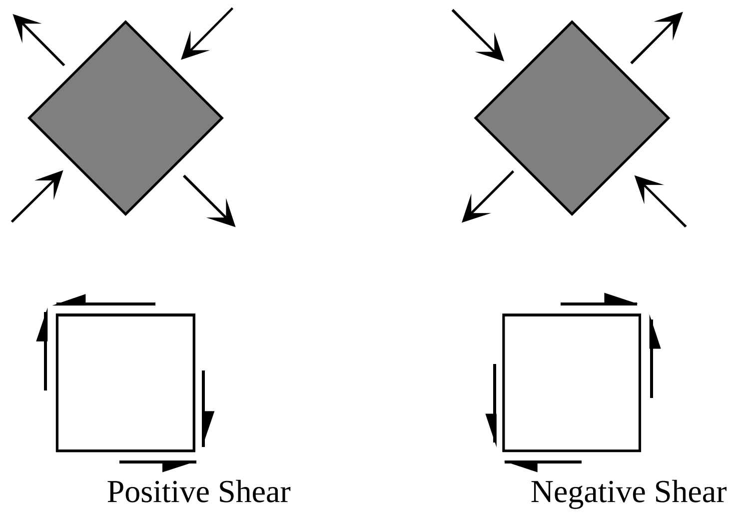

Figure 9-4. Comparison of off-axis positive and negative shear for stress.

compression and the matrix and fiber-matrix interface in tension. Negative shear places the fibers in tension and the matrix and fiber-matrix interface in compression. It is clear that the composite failure mechanisms will be different for the two directions of shear.

BIAXIAL STRENGTH THEORIES

Stiffness and elastic properties in any direction can be obtained by applying tensor transformations. Tensor transformations of strengths are more difficult since the strength tensor is of higher order than stiffness tensor. There are also numerous failure modes that apply depending on the direction of load relative to the principal material directions. For isotropic materials the failure criterion is usually simply defined as yielding or fracture. For Orthotropic material there are considerably more possibilities.

Maximum stress theory

When stresses resolved on the principal material directions exceed a prescribed value, usually the lamina tensile strength in that direction, then failure has occurred. These prescribed values are Y', Y, X' and X determined from uniaxial tensile and compression tests and S or S' determined from in-plane shear tests (remember $S = S'$). Failure will not occur as long as

$$\begin{aligned} -X' &< \sigma_1 < X \\ -Y' &< \sigma_2 < Y \\ -S' &< \tau_{12} < S \end{aligned} \qquad (9.1)$$

Failure will occur if any or all of the above are violated and there are no interaction between failure modes.

If a stress is applied in an arbitrary direction, *x*, say at θ °, from the fiber direction, as illustrated in Fig. 9-5 then those stresses can be transformed to the principal direction to determine if the failure criteria of Eqn. (9.1) has been exceeded.

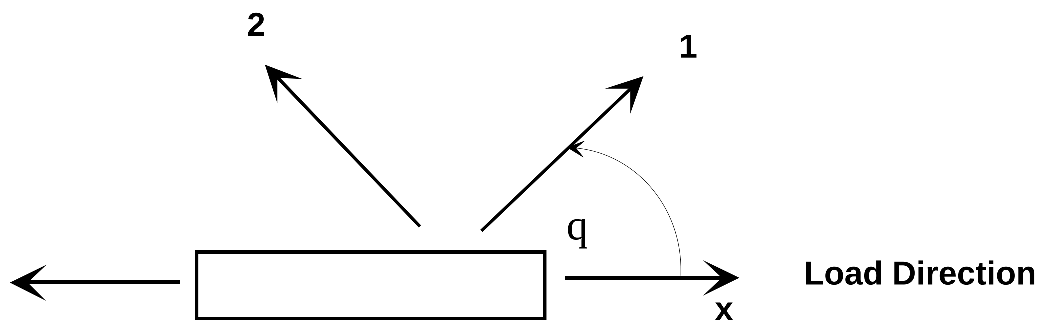

Figure 9-5 Composite lamina loaded in arbitrary direction *x*.

Performing the transformation produces

$$\begin{aligned} \sigma_1 &= \sigma_x \cos^2\theta \\ \sigma_2 &= \sigma_x \sin^2\theta \\ \tau_{12} &= -\sigma_x \cos\theta \sin\theta \end{aligned} \qquad (9.2)$$

Eqn. (9.1) then becomes

$$\begin{aligned} -\frac{X'}{\cos^2\theta} &< \sigma_x < \frac{X}{\cos^2\theta} \\ -\frac{Y'}{\sin^2\theta} &< \sigma_x < \frac{Y}{\sin^2\theta} \\ -\frac{S'}{\cos\theta \sin\theta} &< \sigma_x < \frac{S}{\cos\theta \sin\theta} \end{aligned} \qquad (9.3)$$

Each of these separate criteria can be plotted as a function of fiber orientation θ as is shown in Fig. 9-6. Any stress above the curves exceeds the safe stress for that particular orientation. From this plot it can be seen that five separate criteria must be calculated and examined. At small off-axis angles of the longitudinal strength increase with *2*, hence, the composite is predicted to be stronger in off-axis loading. This prediction is contrary to actual behavior in which the off-axis tensile strength is never greater than on-axis strength. In this plot the individual curves meet at cusps suggesting local maxima at a transition between modes. In reality no such local maxima exists. Examination of experimental data also indicates that this method provides only a rough approximation to the failure stress.

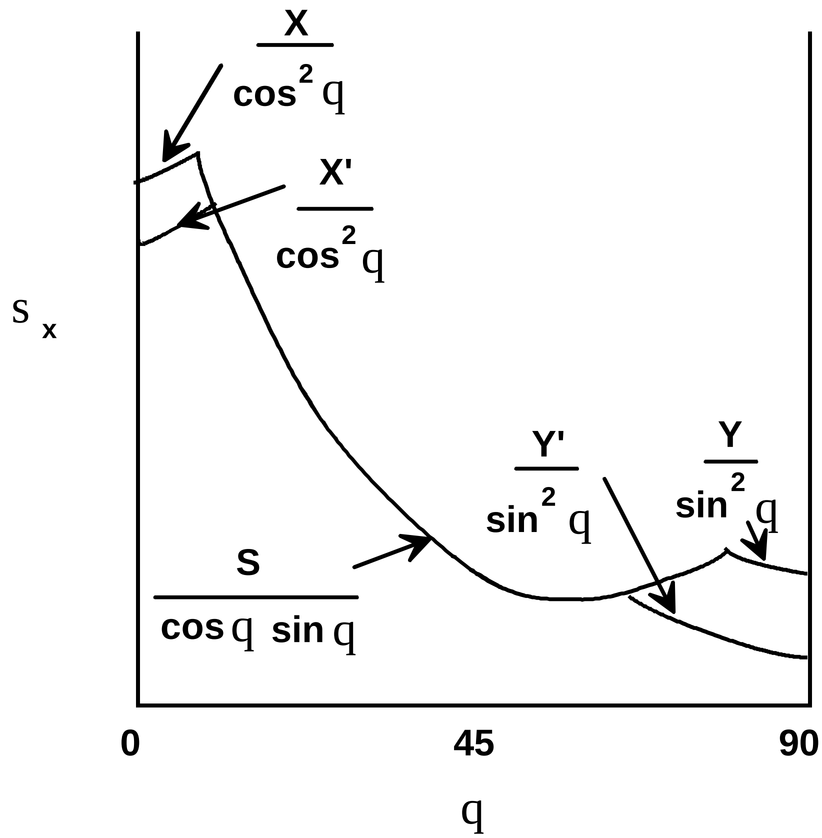

Figure 9-6 Maximum stress theory of lamina failure plotted for off-axis loading

Maximum strain theory

Material or their constituents undergo fracture when a critical separation is produced as a result of stress, therefore a more realistic criterion for failure is strain. This criterion can be written as

$$\begin{aligned} \frac{-X'}{E_1} &< e_1 < \frac{X}{E_1} \\ \frac{-Y'}{E_2} &< e_2 < \frac{Y}{E_2} \\ \frac{-S'}{G_{12}} &< g_{12} < \frac{S}{G_{12}} \end{aligned} \tag{9.4}$$

Using Hooke's Law to express strain in the principal material directions in terms stress in the principal material directions

$$
\begin{aligned}
e_1 &= \frac{s_1}{E_1} - \frac{n_{12} s_2}{E_1} \\
e_2 &= \frac{s_2}{E_2} - \frac{n_{21} s_1}{E_2} \qquad (9.5) \\
g_{12} &= \frac{t_{12}}{G_{12}}
\end{aligned}
$$

Transforming to arbitrary off-axis directions Eqn.(9.5) becomes

$$
\begin{aligned}
e_1 &= \frac{s_x}{E_1}\left(\cos^2 q - n_{12} \sin^2 q\right) \\
e_2 &= \frac{s_x}{E_2}\left(\sin^2 q - n_{21} \cos^2 q\right) \qquad (9.6) \\
g_{12} &= -\frac{s_x}{G_{12}}\left(\cos q \sin q\right)
\end{aligned}
$$

Substituting back into Eqn.(9.4)

$$
\begin{aligned}
-\frac{X'}{\cos^2 q - n_{12} \sin^2 q} &< s_x < \frac{X}{\cos^2 q - n_{12} \sin^2 q} \\
-\frac{Y'}{\sin^2 q - n_{12} \cos^2 q} &< s_x < \frac{Y}{\sin^2 q - n_{12} \cos^2 q} \qquad (9.7) \\
-\frac{S'}{\cos q \sin q} &< s_x < \frac{S}{\cos q \sin q}
\end{aligned}
$$

These values are very similar to those from the maximum stress theory.

STRENGTH THEORIES WITH STRESS INTERACTIONS

In the maximum stress and maximum strain theories the stresses are assumed to act independently. To account for stress interactions various strength tensor theories were developed. The strength tensor in the most general form can be as the polynomial (in contracted notation)

$$
\left(F_i s_i\right)^a + \left(F_{ij} s_i s_j\right)^b + \left(F_{ijk} s_i s_j s_k\right)^g + \ldots = 1 \qquad (9.10)
$$

where $i, j, k = 1, \ldots 6$ and $a = b = 1$.

For the orthotropic case $g = 0$ and the following coefficients are also zero:

$F_4, F_5, F_{6,}\ F_{14}, F_{15}, F_{16,}\ F_{24}, F_{25}, F_{26,}\ F_{34}, F_{35}, F_{36,}\ F_{44}, F_{45}, F_{46}$

These assumptions and simplifications reduce the general polynomial to

$$F_i s_i + F_{ij} s_i s_j = 1 \qquad (9.11)$$

with the following non-zero coefficients: F_1, F_2, F_{12}, F_{21}, F_{22}, F_{66}

Tsai-Hill failure criterion

Using the Von Misses,(distortion energy) criterion Eqn. (9.11) reduces to

$$F_{ij} s_i s_j = 1 \qquad (9.12)$$

which is the basis for the Tsai-Hill criterion. Expanding this tensor gives

$$F_{11} s_1^2 + F_{22} s_2^2 + 2F_{12} s_1 s_2 + F_{66} s_6^2 = 1 \qquad (9.13)$$

The constants F_{11}, F_{22}, F_{12}, and F_{66} can be related to the strength properties of the lamina in the principal material directions by considering uniaxial loading to failure for all three stresses. First consider the case, $s_1 = X$, $s_2 = 0$ and $s_6 = 0$, then Eqn (9.13) reduces to

$$F_{11} X^2 = 1 \qquad (9.14)$$

Next consider the case $s_2 = Y$, $s_1 = 0$ and $s_6 = 0$, then Eqn. (9.13) becomes

$$F_{22} Y^2 = 1 \qquad (9.15)$$

The last case to consider is $s_6 = S$, and $s_1 = s_2 = 0$, then Eqn. (9.13) becomes

$$F_{66} S^2 = 1 \qquad (9.16)$$

To find F_{12} apply a balanced biaxial stress, s_1 and s_2 equal to the longitudinal tensile strength, X and apply no shear stress $s_6 = 0$ then Eqn. (9.13) becomes

$$2F_{12} X^2 = -1 \qquad (9.17)$$

The strength coefficients can now be obtained from Eqns, (9.14),(9.15), (9.16) and (9.17)

$$F_{11} = \frac{1}{X^2}$$
$$F_{22} = \frac{1}{Y^2}$$
$$F_{66} = \frac{1}{S^2} \qquad (9.18)$$
$$F_{12} = -\frac{1}{2X^2}$$

From Eqn. (913) the Tsai-Hill criterion can now be written

$$\frac{s_1^2}{X^2} + \frac{s_2^2}{Y^2} - \frac{s_1 s_2}{X^2} + \frac{s_6^2}{S^2} = 1 \qquad (9.19)$$

Using Eqn. (9.2) the Tsai-Hill criterion predicts the failure stress, s_x for off-axis loading

$$s_x = \sqrt{\frac{1}{\frac{\cos^4 q}{X^2} + \frac{\sin^4 q}{Y^2} + \cos^2 q \sin^2 q \left(\frac{1}{S^2} - \frac{1}{X^2} \right)}} \qquad (9.20)$$

Fig. 9-7 is a plot of Eqn. (9.20) for composite with a tensile, compressive and shear strengths of 80 ksi, 15 ksi and 19 ksi respectively in the principal material directions.

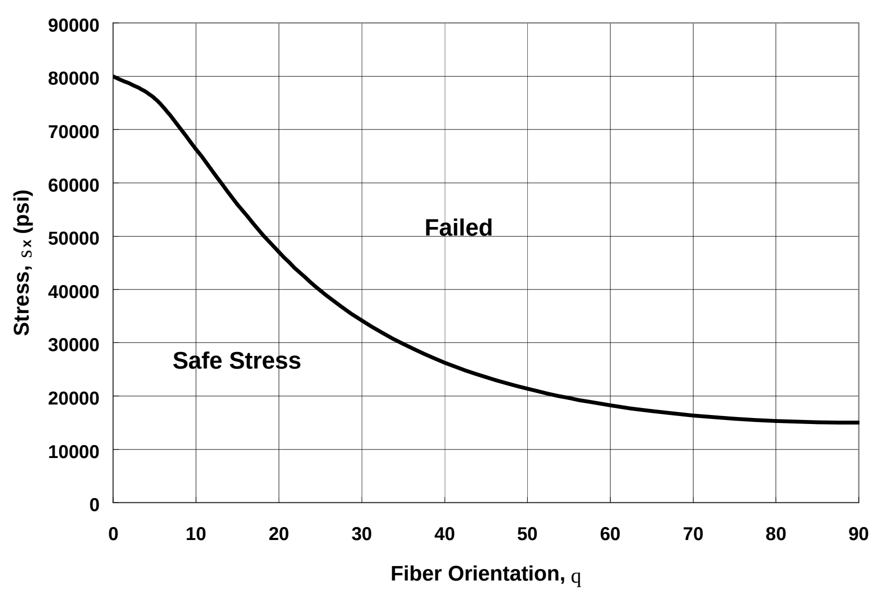

Figure 9-7 Predicted failure stress, s_x using Tsai-Hill criterion

The failure stress in terms of s_y and t_{xy} can be similarly determined. This plots agrees quite reasonably with experimental data. The failure stress curve is smooth and continuous and always decreases with increasing degree of off-axis angle. This criterion accounts for the interaction between stress and reduces to the correct form for isotropic material. This criterion, however, does not address compressive failure. A modification to the Tsai-Hill criterion, now referred to as the Tsai-Wu criterion was developed to correct this limitation.

Tsai-Wu failure criterion

The Tsai-Wu criterion uses the tensor form of Eqn. (9.11) expanded to

$$F_6 s_6 + F_1 s_1 + F_2 s_2 + 2F_{12} s_1 s_2 + F_{11} s_1^2 + F_{22} s_2^2 + F_{66} s_6^2 = 1 \qquad (9.21)$$

As was done to derive the constants for the Tsai-Hill criterion we apply a stress only in the longitudinal fiber direction, F_1, that is equal to the longitudinal tensile strength *X* and the apply no transverse, F_2, and shear, F_6, stresses then Eqn (9.21) reduces to

$$F_1 X + F_{11} X^2 = 1 \qquad (9.22)$$

We can also set the applied longitudinal stress, F_1, to the compressive longitudinal strength, *X'* (the value of *X'* is positive but has a negative sign assigned to it) and the transverse, F_2 and shear stress F_6 to zero and get

$$-F_1 X' + F_{11} X'^2 = 1 \qquad (9.23)$$

Using Eqns. (9.22) and (.23) the coefficients F_1 and F_{11} can be found

$$F_1 = \frac{1}{X} - \frac{1}{X'} \qquad (9.24a)$$

$$F_{11} = \frac{1}{XX'} \qquad (9.24b)$$

Likewise the following coefficients can be found

$$F_2 = \frac{1}{Y} - \frac{1}{Y'} \qquad (9.24c)$$

$$F_{22} = \frac{1}{YY'} \qquad (9.24d)$$

$$F_{66} = \frac{1}{S^2} \qquad (9.24e)$$

Applying the measured biaxial tensile strength of the lamina, B to s_1 and s_2 only the coefficient F_{12} becomes

$$F_{12} = \frac{1}{2B^2}\left[1 - B\left(\frac{1}{X} - \frac{1}{X'} + \frac{1}{Y} - \frac{1}{Y'}\right) - B^2\left(\frac{1}{XX'} + \frac{1}{YY'}\right)\right] \qquad (9.24f)$$

If biaxial strength data is not available F_{12} can be found empirically as

$$F_{12} = F_{12}^* \sqrt{F_{11} F_{22}}$$

where F_{12}^* is between -0.5 and 0.

Using F_{12}^* = -0.5 the usual form of the Tsai-Wu criterion is

$$\left(\frac{1}{X} - \frac{1}{X'}\right)s_1 + \left(\frac{1}{Y} - \frac{1}{Y'}\right)s_2 + \frac{1}{XX'}s_1^2 + \frac{1}{YY'}s_2^2 - \frac{1}{2}\sqrt{\frac{1}{XX'YY'}}s_1 s_2 + \frac{1}{S^2}s_6^2 = 1$$
(9.25)

<u>FAILURE ENVELOPES STRESS SPACE</u>

Plotting the biaxial stresses that result in failure produced the so-called *failure envelope.* For isotropic brittle materials the failure is independent of the direction of loading and the failure envelope can be defined for the principal stresses, s_I and s_{II} and will occur if either stress exceeds the fracture strength, s_{UTS} of the material. The maximum stress theory would satisfactorily failure of such a material. The failure envelope for this case is shown in Fig. 9-8. Stress combinations that fall outside the envelope constitute failure.

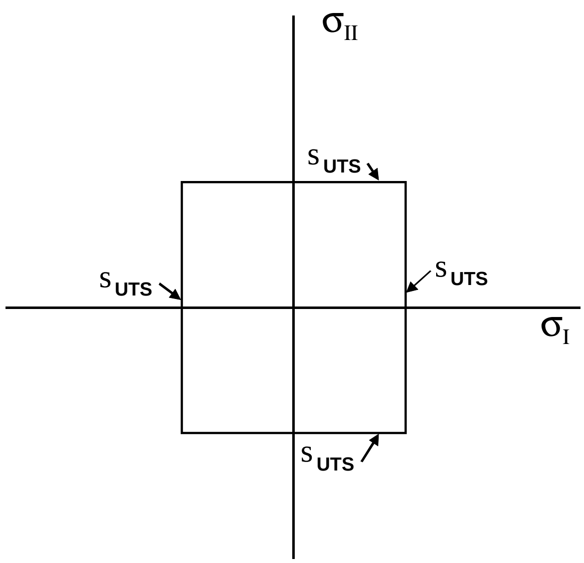

Figure 9-8 Failure envelope for isotropic brittle material

For isotropic ductile materials, such as most metals, failure is defined when yielding occurs, s_{yield}. In that case either the maximum shear stress theory (Tresca) or the distortion energy theory (Von Mises-Hencky) is satisfactory. Fig. 9-9 shows the failure envelope for materials that are satisfactorily represented by the Tresca or Von Mises-Hencky criteria.

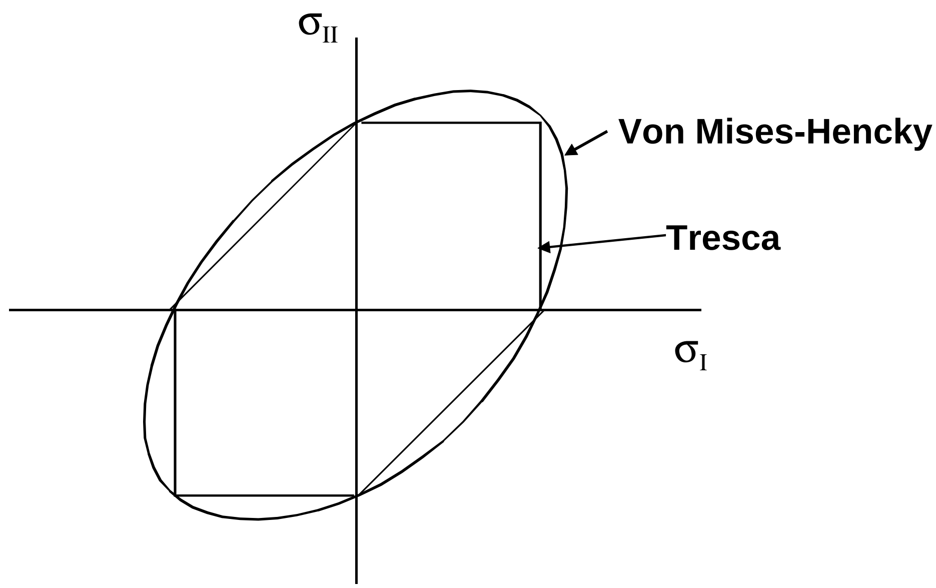

Figure 9-9 Failure envelopes for isotropic ductile materials

The failure envelope for orthotropic material obeying the maximum stress theory, Eqn. (9.1) is shown in Fig. 9-10. In this case failure is defined in terms of the stresses in the principal material directions. Unlike isotopic materials the failure stresses are different in the different principal material directions and in the tensile and compressive directions due to different failure mechanisms applying. The safe stresses fall within the envelope defined by the four equations

$$\begin{aligned} \sigma_1 &= X \\ \sigma_1 &= X' \\ \sigma_2 &= Y \\ \sigma_2 &= Y' \end{aligned} \qquad (9.26)$$

The failure envelope for the maximum strain theory, Eqn. (9.4), shown in Fig. 9-11, is composed of the four equations

$$\begin{aligned} \sigma_1 &= X + \sigma_2 \nu_{12} \\ \sigma_1 &= X' + \sigma_2 \nu_{12} \\ \sigma_2 &= Y + \sigma_1 \nu_{21} \\ \sigma_2 &= Y' + \sigma_1 \nu_{21} \end{aligned} \qquad (9.27)$$

The maximum strain theory recognizes that strain controls fracture and takes into account the combined stresses in the principal material directions.

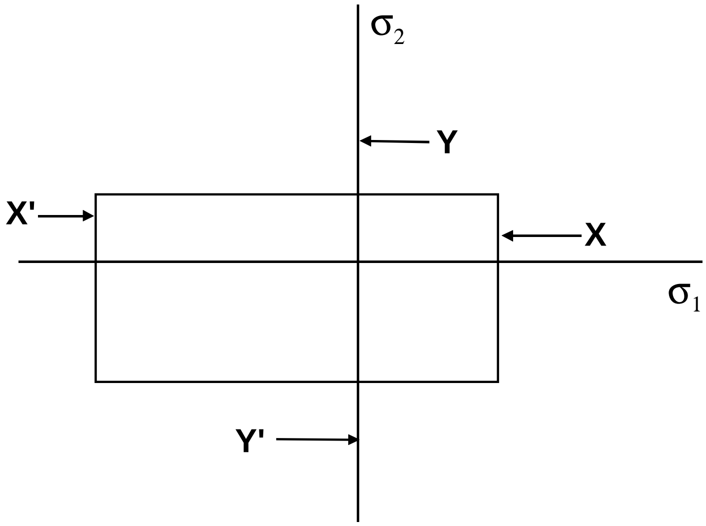

Figure 9-10. Failure envelope for orthotopic material by the maximum stress theory

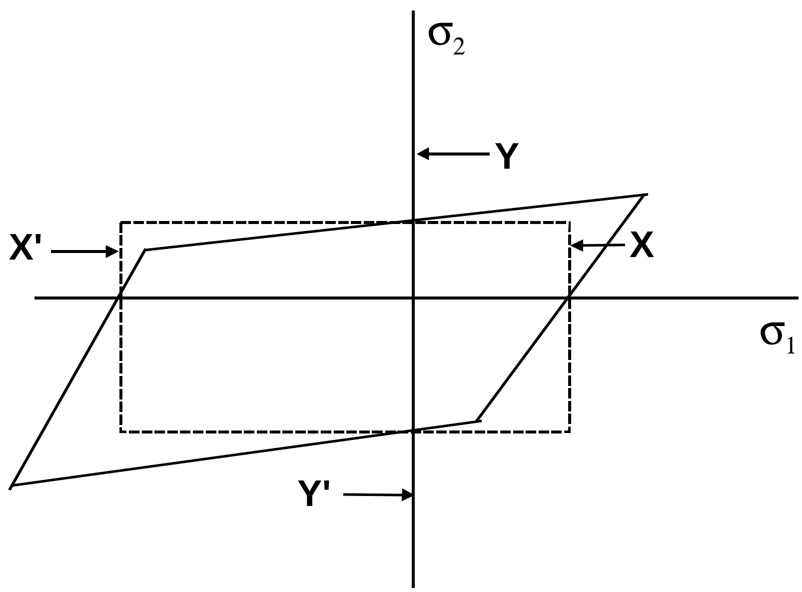

Figure 9-11. Failure envelope for orthotopic material by the maximum strain theory

The maximum stress failure envelope is shown as the dashed line in this figure for comparison. The maximum strain failure envelope intersects the maximum stress failure envelope at the strength values where one of the stresses in the principal material directions is zero.

For the Tsai-Hill or Tsai-Wu criteria the failure envelope is an ellipse. For the Tsai-Wu criterion Eqn. (9.21) and be rearranged in quadratic form in terms of s_1 as the dependent variable

$$s_1^2 F_{11} + s_1 \left(F_1 + 2 s_2 F_{12} \right) + \left(s_2 F_2 + s_2^2 F_{22} + s_6^2 F_{66} - 1 \right) = 0 \qquad (9.26)$$

Plotting σ_1 vs. σ_2 of Eqn.(9.26) produces the *failure ellipse* in stress space, shown in Fig 9-12. As with the other failure envelopes, points that lie inside the ellipse constitute a safe stress range and points that lie outside constitute failure.

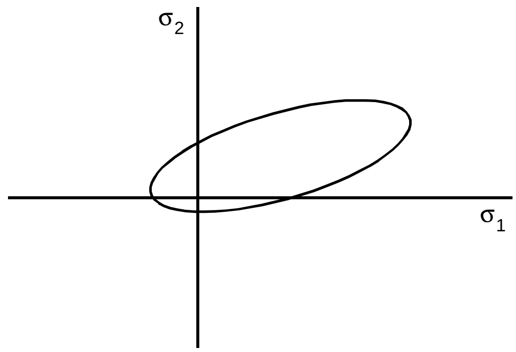

Figure 9.12 Failure ellipse for Tsai-Wu criterion in stress space

FAILURE ELLIPSE IN STRAIN SPACE

For highly anisotropic composites the ellipse is exaggerated in the major axis direction (i.e. cigar shaped) and then plotting the ellipse in strain space is preferred. Hooke's law is used to the failure criterion in strain space

$$\sigma_i = Q_{ij}\varepsilon_j \tag{9.27}$$

Substituting Eqn. (9.27) into the stress criterion Eqn. (9.11) gives

$$F_{ij}Q_{ik}Q_{jl}\varepsilon_k\varepsilon_l + F_iQ_{ij}\varepsilon_j = 1 \tag{9.28}$$

using the following notation: $F_{ij}Q_{ik}Q_{jl} = H_{kl}$ and $F_iQ_{ij} = H_i$ Eqn. (9.28) can be written as

$$H_{ij}\varepsilon_i\varepsilon_j + H_i\varepsilon_i = 1 \tag{9.29}$$

In expanded form Eqn. (9.29) is

$$H_1\varepsilon_1 + H_2\varepsilon_2 + H_{11}\varepsilon_1^2 + 2H_{12}\varepsilon_1\varepsilon_2 + H_{22}\varepsilon_2^2 + H_{66}\varepsilon_6^2 = 1 \tag{9.30}$$

where

$$H_{11} = F_{11}Q_{11}^2 + 2F_{12}Q_{11}Q_{12} + F_{22}Q_{12}^2$$
$$H_{22} = F_{22}Q_{22}^2 + 2F_{12}Q_{22}Q_{12} + F_{11}Q_{12}^2$$
$$H_{12} = F_{11}Q_{11}Q_{12} + F_{12}\left(Q_{11}Q_{22} + Q_{12}^2\right) + F_{22}Q_{12}Q_{22}$$
$$H_{66} = F_{66}Q_{66}^2$$
$$H_1 = F_1Q_{11} + F_2Q_{12}$$
$$H_2 = F_1Q_{12} + F_2Q_{22}$$

Solving Eqn. (9.30) for e_1 produces the quadratic equation

$$e_1^2H_{11} + e_1\left(H_1 + 2e_2H_{12}\right) + \left(e_2H_2 + e_2^2H_{22} + e_6^2H_{66} - 1\right) = 0 \qquad (9.31)$$

Plotting e_1 vs. e_2 of Eqn (9.31) produces the failure ellipse in strain space of Fig.9-13. Points that lie inside the ellipse constitutes a safe strain range and points outside constitutes failure. The aspect ratio of the strain space failure ellipse is smaller than the aspect ratio for the stress space ellipse. It may be noticed that the shear stress, s_6 or shear strain, e_6 is a constant in these equations. The maximum size of the ellipse occurs for $s_6 = 0$ or $e_6 = 0$. When $s_6 = S$ or $e_6 =$ the failure strain in shear the ellipse shrinks to a point.

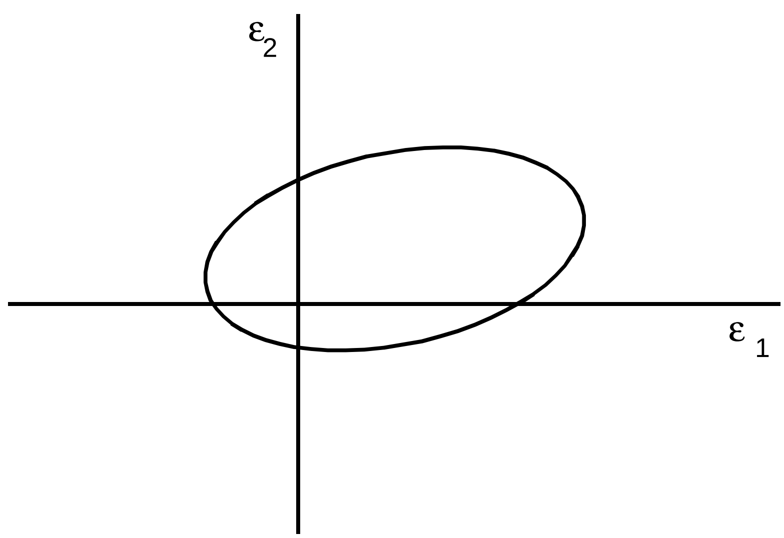

Figure 9.12 Failure ellipse for Tsai-Wu criterion in stress space

USING THE FAILURE CRITERIA AS A DESIGN TOOL

The ratios, $\frac{X}{\sigma_x}, \frac{Y}{\sigma_y}, \frac{S}{\tau_{xy}}$ are referred to a "strength ratios", R_x, $R_{y,}$ R_S respectively. They can be regarded as factors of safety. X, Y and S are fixed for a given composite therefore will determine the safety factor. The strength tensor in terms of the strength ratio, *R*, is written as

$$\left(F_i\sigma_i\right)R+\left(F_{ij}\sigma_i\sigma_j\right)R^2=1 \qquad (9.32)$$

or on strain space as

$$\left(H_i\varepsilon_i\right)R+\left(H_{ij}\varepsilon_i\varepsilon_j\right)R^2=1 \qquad (9.33)$$

These can be solved as a quadratics roots R_1 or R_+ and R_2 or R_- from the solutions

$$\frac{-b+\sqrt{b^2-4ac}}{2a} \quad \text{and} \quad \left|\frac{-b-\sqrt{b^2-4ac}}{2a}\right|$$

The root R_2 or R_- from the conjugate solution predicts the factors of safety if all of the stresses are reversed in sign.

Other strength theories can be found in the Addendum to this chapter.

ADDENDUM TO CHAPTER 9

In addition to Tsai-Hill and Tsai-Wu there are numerous other stress interaction strength criteria. These are listed in Table 9A-1.

Table 9A-1 Stress interaction strength criteria

Theory	Tensor	F_1	F_2	F_{11}	F_{22}	F_{12}	F_{66}
Ashkenazi	$F_{ij} s_i s_j = 1$			$\frac{1}{X^2}$	$\frac{1}{Y^2}$	$\frac{1}{2}\left(\frac{4}{U^2} - \frac{1}{X^2} - \frac{1}{Y^2} - \frac{1}{S^2}\right)$	$\frac{1}{S^2}$
Cowin	$F_i s_i + F_{ij} s_i s_j = 1$	$\frac{1}{X} - \frac{1}{X'}$	$\frac{1}{Y} - \frac{1}{Y'}$	$\frac{1}{XX'}$	$\frac{1}{YY'}$	$\sqrt{F_{11}F_{22}} - \frac{1}{S^2}$	$\frac{1}{S^2}$
Fischer	$F_{ij} s_i s_j = 1$			$\frac{1}{X^2}$	$\frac{1}{Y^2}$	$-\frac{K}{2XX'}$	$\frac{1}{S^2}$
Hoffman	$F_i s_i + F_{ij} s_i s_j = 1$	$\frac{1}{X} - \frac{1}{X'}$	$\frac{1}{Y} - \frac{1}{Y'}$	$\frac{1}{XX'}$	$\frac{1}{YY'}$	$-\frac{1}{2XX'}$	$\frac{1}{S^2}$
Malmeister	$F_i s_i + F_{ij} s_i s_j = 1$	$\frac{1}{X} - \frac{1}{X'}$	$\frac{1}{Y} - \frac{1}{Y'}$	$\frac{1}{XX'}$	$\frac{1}{YY'}$	$\frac{1}{XX'} - \frac{XX' - S\left[X' - X - X'\left(X/Y\right) + Y\right]}{2S^2 XX'}$	$\frac{1}{S^2}$
Marin	$F_i s_i + F_{ij} s_i s_j = 1$	$\frac{1}{X} - \frac{1}{X'}$	$\frac{1}{Y} - \frac{1}{Y'}$	$\frac{1}{XX'}$	$\frac{1}{YY'}$	$-\frac{1}{2XY}$	$\frac{1}{S^2}$
Tsai-Hill	$F_{ij} s_i s_j = 1$			$\frac{1}{X^2}$	$\frac{1}{Y^2}$	$-\frac{1}{2X^2}$	$\frac{1}{S^2}$
Tsai-Wu	$F_i s_i + F_{ij} s_i s_j = 1$	$\frac{1}{X} - \frac{1}{X'}$	$\frac{1}{Y} - \frac{1}{Y'}$	$\frac{1}{XX'}$	$\frac{1}{YY'}$	$F_{12}^{*}\sqrt{F_{11}F_{22}}$	$\frac{1}{S^2}$