7 Short Fiber Composites

Stress on a discontinuous fiber

In a composite where the fibers are continuous in one direction, i.e. the fibers do not end inside the composite, the stress on the fiber can be predicted easily by the rule-of-mixtures. If the fibers are shorter than the length of the composite they are discontinuous in the composite and the rule-of-mixtures may not necessarily accurately predict fiber stress. In that case the stress on the fiber will depend on the length of the fibers, the elastic and plastic properties fibers and matrix and the shear strength of the matrix or the fiber-matrix interfacial strength, t_i.

Elastic fiber – rigid/perfectly plastic matrix

To calculate the stress on the fiber some assumptions concerning the elastic-plastic properties of the constituents are required. For the simplest model, the fiber is assumed to be perfectly elastic and the matrix is rigid/perfectly plastic. The fiber is assumed to behave as a linear elastic solid while the matrix is assumed to have infinite stiffness up to the shear strength of the matrix at which point the matrix yields with no work hardening. These conditions are shown in Fig. 7-1.

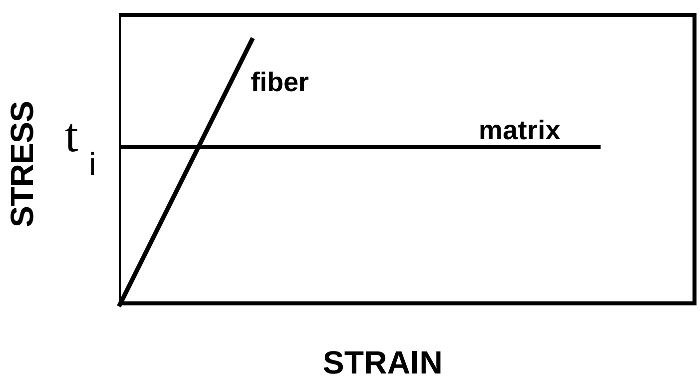

Figure 7-1. Stress-strain behavior of elastic fiber and rigid/perfectly plastic matrix.

With this material model all forces on the fiber are transmitted from the matrix through the surface of the fiber. Fig. 7-2 shows a sample fiber in the matrix before and after loading. The fiber extends elastically while the matrix slides plastically along the fiber at a constant stress. The matrix yield strength in shear controls the maximum stress that the fiber can experience. The higher the matrix yield strength the greater is the stress that can be transferred to the fiber. If the fibers and matrix are not well bonded then the stress in the matrix can only be transferred to the fiber by the frictional force between the fiber and the matrix. In this case the coefficient of friction between fiber and matrix along with normal force the matrix imposes on the fiber that control the maximum stress that can be transmitted to the fiber. Using this logic it is clear that if the coefficient of expansion in the radial direction of the fiber is greater than the coefficient of expansion of the matrix the fiber will separate completely from the matrix during cool-down from the fabrication temperature and no stress will be transferred to the fiber. In this case the composite will have both a lower strength and stiffness than the matrix alone.

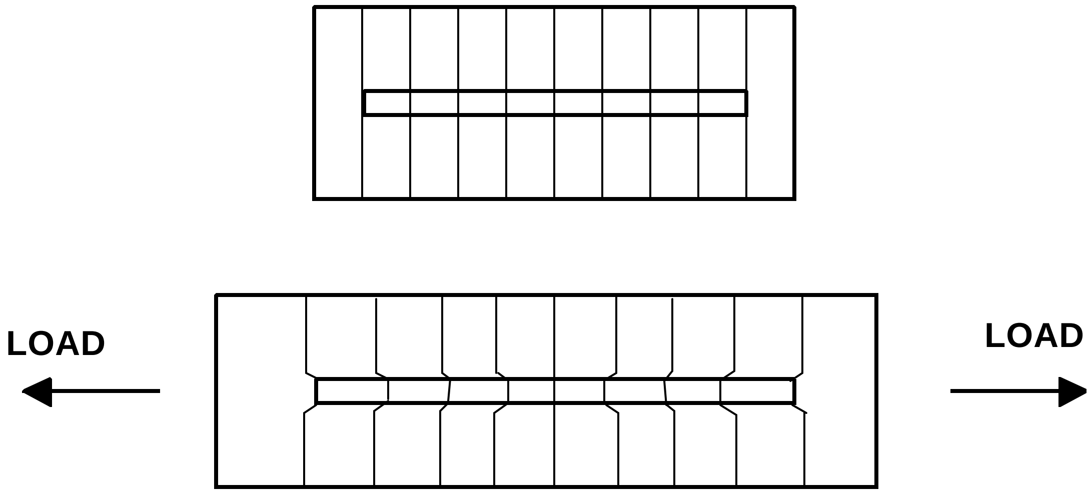

Figure 7-2 Sample discontinuous fiber in matrix. (Upper) Unloaded. (Lower) Loaded

To determine the stresses transferred to the fiber in a discontinuous composite consider all the forces in one half a fiber of length, ℓ and radius, r shown in Fig.7-3. The sum of forces in element dx consist of end forces $\sigma_f A_c$ and shear forces $\tau_i A_s$, where A_c is the cross-sectional area of the fiber and A_s is the surface area of the fiber.

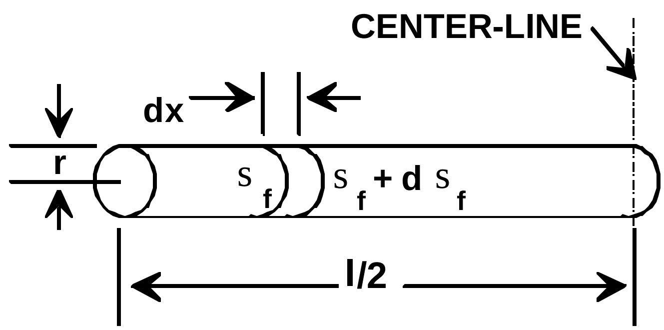

Figure 7-3 Half of a discontinuous fiber of length ℓ.

The summation of forces in element dx gives

$$\sigma_f \pi r^2 + \tau_i 2\pi r\, dx = \sigma_f \pi r^2 + \pi r^2\, d\sigma_f$$

which simplifies to

$$\tau_i 2\, dx = r\, d\sigma_f$$

integrating

$$\int_{\sigma_{f0}}^{\sigma_{f\max}} d\sigma_f = \int_0^{l/2} \frac{2\tau_i}{r}\, dx \qquad (7.1)$$

where s_{f_0} is the stress due to loading on the end cross section of the fiber and $s_{f_{max}}$ is the stress at the mid-point (center-line) of the fiber which terns out to be the largest stress in the fiber. The stress due to loading on the end cross section of the fiber can be neglected hence the evaluation of the integral Eqn.(7.1) gives

$$s_{f\,max} = \frac{t_i \ell}{r} \qquad (7.2)$$

Thus the stress on the fiber is a straight line starting from 0 at the end of the fiber and reaching a maximum at the center of the fiber. Integrating from the center of the fiber to the other end results on the stress decreasing from the maximum at the mid-point to 0 at the other end. The tensile the fiber is shown in Fig. 7-4. The shear stress along the fiber is constant as shown in Fig. 7-5.

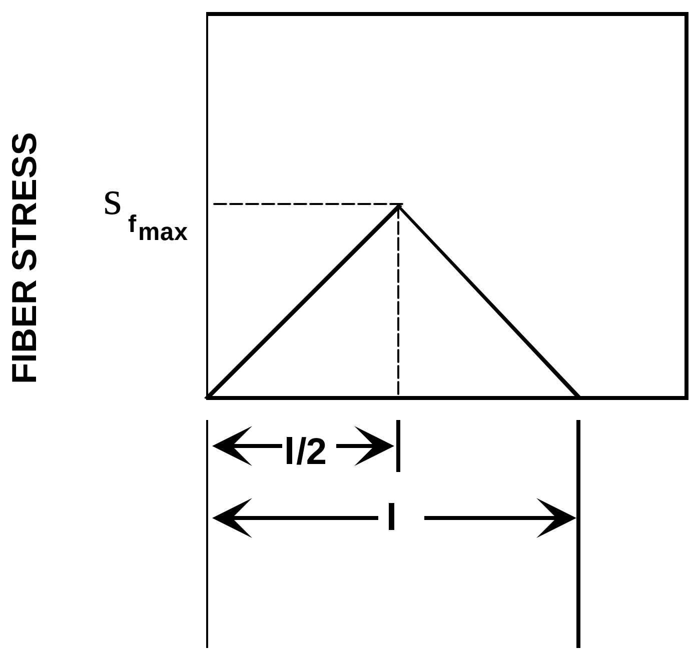

Figure 7-4 Stress in discontinuous fiber along its length

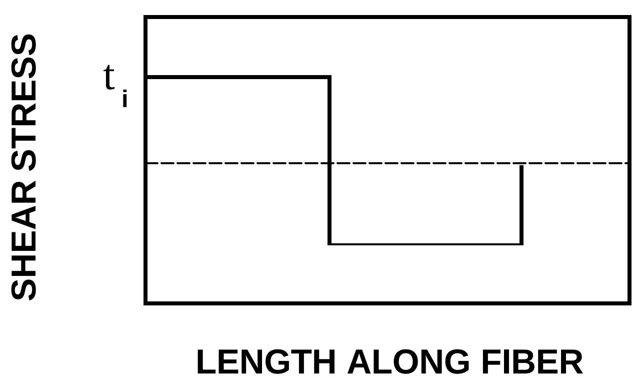

Figure 7-5 Shear stress on surface of discontinuous fiber

Ineffective length

If the fibers in a composite are long but still discontinuous and the composite is loaded to a stress, s_c less than the strength of the composite, s_{cu} then the maximum stress that can be generated in the fiber is

$$s_{f\max} = \frac{E_f}{E_c} s_c \qquad (7.3)$$

The maximum stress will occur along the length of the fiber except over a length $\ell_t/2$ at each end of the fiber where the stress will diminish to 0, as seen in Fig.7-6. The total length in the fiber that carries a stress below the maximum is ℓ_t, hence it is called the ineffective length. The length of the fiber that carries the stress predicted by the rule-of-mixtures is $\ell - \ell_t$. Combining Eqns. (7.2) and (7.3) and solving for ℓ yields the ineffective length

$$\ell_t = r \frac{E_f}{E_c} \frac{s_c}{t_i} \qquad (7.4)$$

The ineffective length is therefore not only a characteristic of the fiber and matrix but also depends the amount of applied stress to the composite. The shorter the ineffective length, the greater is the efficiency of the fiber in carrying stress. An efficient composite is one that has a small diameter and high interfacial shear strength.

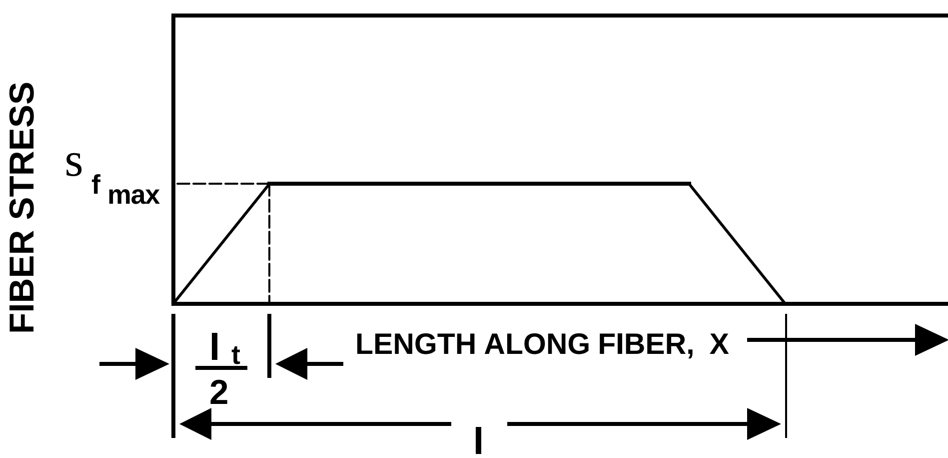

Figure 7-6 Stress in a long discontinuous fiber.

If the composite contains fiber lengths, ℓ less than ℓ_t then the fibers cannot be loaded to their maximum potential hence $s_f < s_{f\max}$

$$s_f < \frac{E_f}{E_c} s_c$$

This situation is illustrated in Fig. 7-7.

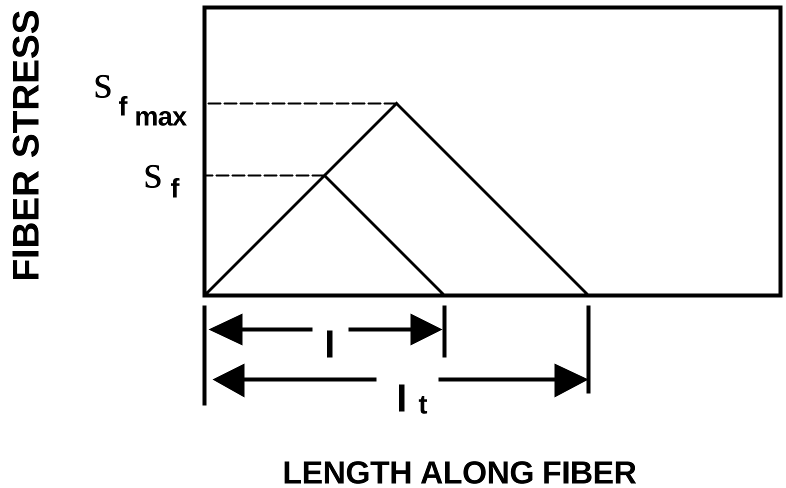

Figure 7-7. Stress distribution on discontinuous fiber of length less than ℓ_t

Critical length

If the composite is loaded to the ultimate or breaking strength, s_{cu}, the ineffective length becomes the *critical length*, ℓ_c and the discontinuous fibers in the composite can be loaded to failure hence,

$$s_{fu} = \frac{E_f}{E_c} s_{cu}$$

and the critical length is given as

$$\ell_c = \frac{d s_{fu}}{2t_i}$$

where *d* is the fiber diameter. Fig.7-8 compares the stress distribution along the length of discontinuous fibers in a composite based on fiber length. The ineffective length, ℓ_t is defined uniquely for any arbitrary stress applied to the composite while the critical length, ℓ_c is a properties of the fiber strength.

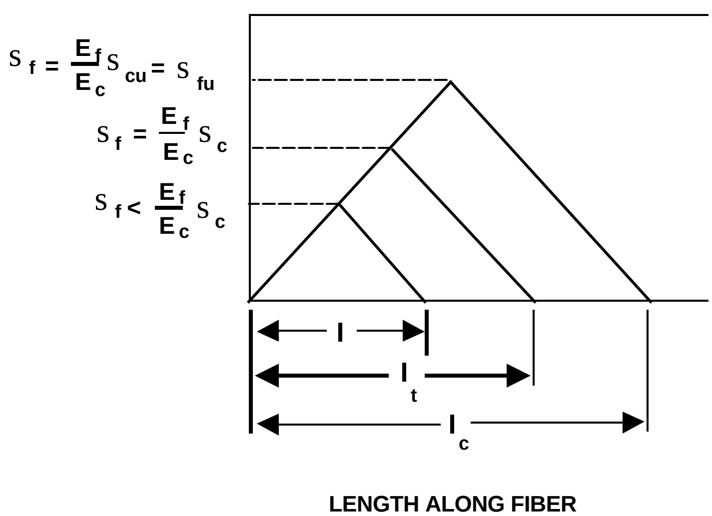

Figure 7-8 Comparison of stresses in discontinuous fibers at various lengths.

Stress transfer for elastic matrix and elastic fiber

The assumption that the matrix behaves as a rigid perfectly plastic solid does not apply to many common composite materials such as most polymer matrix composites. A more realistic model would assume that both the fiber and the matrix are elastic. Fig.7-9 shows the strain fields around a discontinuous elastic fiber in an elastic matrix.

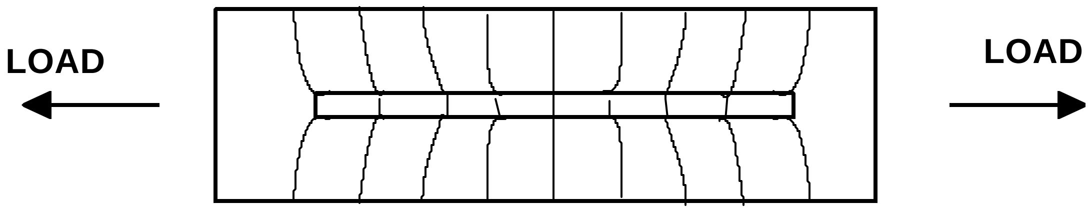

Figure 7-9. Representation of the stress field in the matrix on an elastic discontinuous fiber in an elastic matrix.

In the elastic stress transfer model there is no yielding at the interface between fiber and matrix. The strain in the matrix and in the composites is considered equal, $e_m = e_c$, however the strain in the fiber is clearly less than that of the composite. If we consider a cylindrical composite in which the fiber is in the center of the cylinder, surrounded by matrix, the radial distance from the center of the fiber is given by *z,* the circumferential angle is q, and the longitudinal direction is *x*. Fig. 7-10 is the end view of such a composite looking in the direction parallel to the fiber. The displacement of the matrix

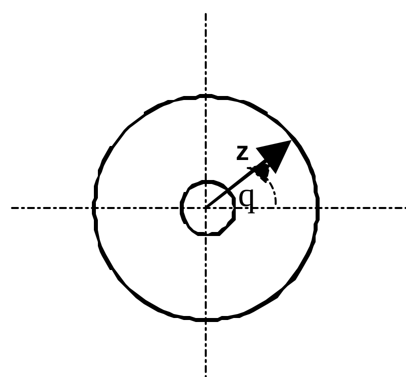

Figure 7-10. End view of composite for elastic fiber and matrix model.

The matrix displacement *w* in the fiber direction is assumed constant for any distance z from the fiber over all possible angles, q, hence

$$w = f(z)$$

We can also assume that the shear stress, t, is also independent of q

$$t = f(z)$$

The matrix displacement at the surface of the fiber is equal to the fiber displacement, u_f and matrix displacement at the outside surface on the composite cylinder it is u_R. Fig.7-11 shows the elastic matrix displacement around an elastic fiber.

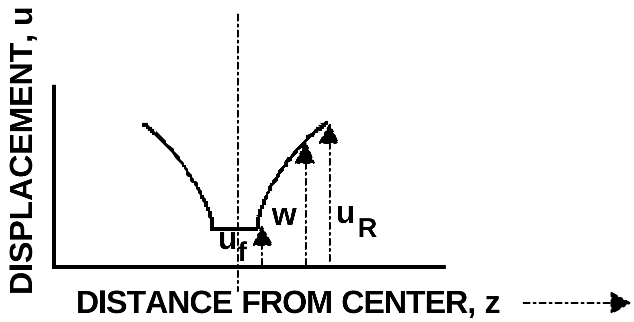

Figure 7-11 Matrix displacement around a fiber in a cylindrical composite

To apply this model composites with multiple fibers of radius, *r*, assume that there is a uniform and regular array of fibers in the matrix. An element *dx* along the longitudinal direction is shown in Fig. 7-12. The radius of the unit cell in the multi-fiber composite is *R*. The shear stress on the fiber-matrix interface is τ_e.

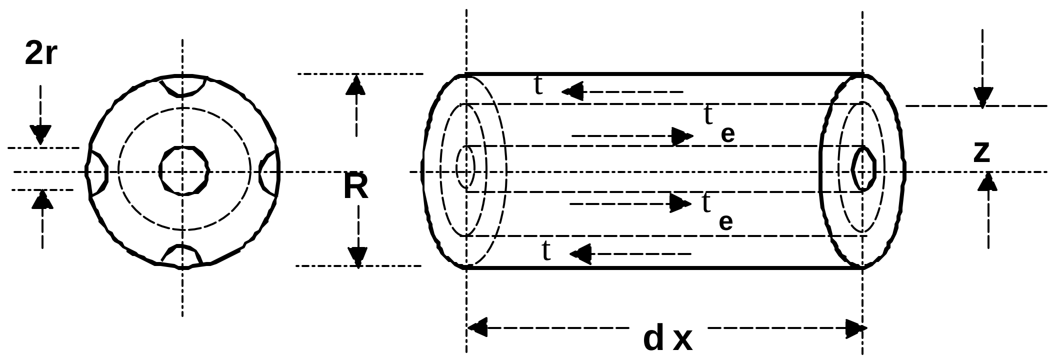

Figure 7-12 Multi-fiber model showing element along longitudinal direction

Summing the shear forces at the fiber surface and at a distance *z* form the center of the fiber gives

$$2\pi z\tau\, dx = 2\pi r\tau_e\, dx$$

which reduces to

$$\tau = \frac{r}{z}\tau_e$$

Expressing the shear stress in terms of shear strain and shear modulus of the matrix

$$\frac{du}{dz} = \frac{r\tau_e}{zG_m}$$

To solve for the displacement

$$\int_{u_f}^{u_R} du = \frac{rt_e}{G_m}\int_r^R \frac{dz}{z}$$

which evaluates to

$$u_R - u_f = \frac{rt_e}{G_m}\ln\frac{R}{r} \qquad (7.5)$$

The ratio *R/r* depends upon the arrangement of the fibers which can be expressed as the packing factor, P_f.

For square array of fibers

$$\ln\frac{R}{r} = \frac{1}{2}\ln\left(\frac{p}{V_f}\right)$$

For hexagonal array of fibers

$$\ln\frac{R}{r} = \frac{1}{2}\ln\left(\frac{2p}{\sqrt{3}V_f}\right)$$

For a general arrangement

$$\ln\frac{R}{r} = \frac{1}{2}\ln\left(\frac{P_f}{V_f}\right) \qquad (7.6)$$

Substituting Eqn,(7.6) in to Eqn. (7.5) and solving for t_e

$$t_e = \frac{E_m(u_R - u_f)}{(1+n_m)r\ln\left(\frac{P_f}{V_f}\right)} \qquad (7.7)$$

Taking the center of the fiber as the origin the stress transferred to the fiber can be written as

$$\frac{ds_f}{dx} = -\frac{2t_e}{r} \qquad (7.8)$$

Combining Eqns. (7.7) and (7.8)

$$\frac{ds_f}{dx} = -\frac{2E_m(u_R - u_f)}{(1+n_m)r^2\ln\left(\frac{P_f}{V_f}\right)} \qquad (7.9)$$

Differentiating Eqn. (7.9)

$$\frac{d^2s_f}{dx^2} = \frac{n^2}{r^2}\left(s_f - E_f e_1\right) \qquad (7.10)$$

where

$$n^2 = \frac{2E_m}{E_f\left(1+n_m\right)\ln\left(\frac{P_f}{V_f}\right)} \tag{7.11}$$

and e_1 is the apparent strain of the fiber.

The solution of Eqn(7.10) at the appropriate boundary condition gives

$$s_f = E_f\, e_1 \left\{ 1 - \frac{\cosh\left(\frac{nx}{r}\right)}{\cosh\left(\frac{n\ell}{2r}\right)} \right\} \tag{7.12}$$

where ℓ is the length of the fiber.

Similarly the interfacial shear stress can be found

$$t_e = \frac{\frac{1}{2} nE_f e_1 \sinh\left(\frac{nx}{r}\right)}{\cosh\left(\frac{n\ell}{2r}\right)} \tag{7.13}$$

The Eqns. (7.12) and (7.13) are plotted in Fig. 7-13.

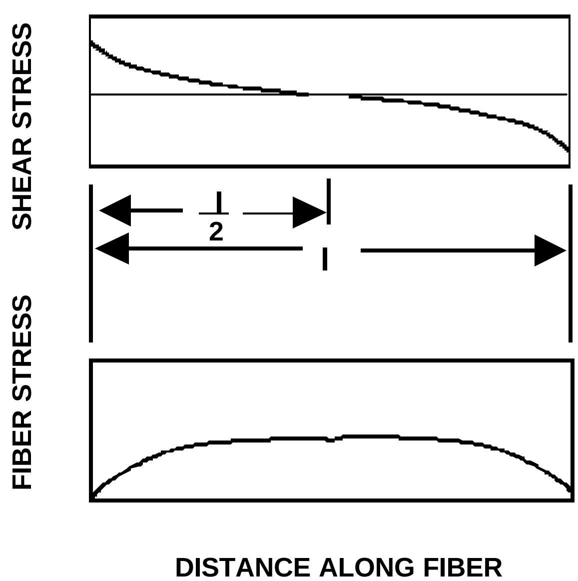

Figure 7-13. Shear stress and fiber stress distribution along fiber in composite with elastic fiber and elastic matrix

We can now compare the fiber and shear stress for different fiber and matrix behavior in Fig. 7-14. The two cases just discussed are compared to the case where the matrix has elastic plastic behavior. This would be the case for a metal matrix composite.

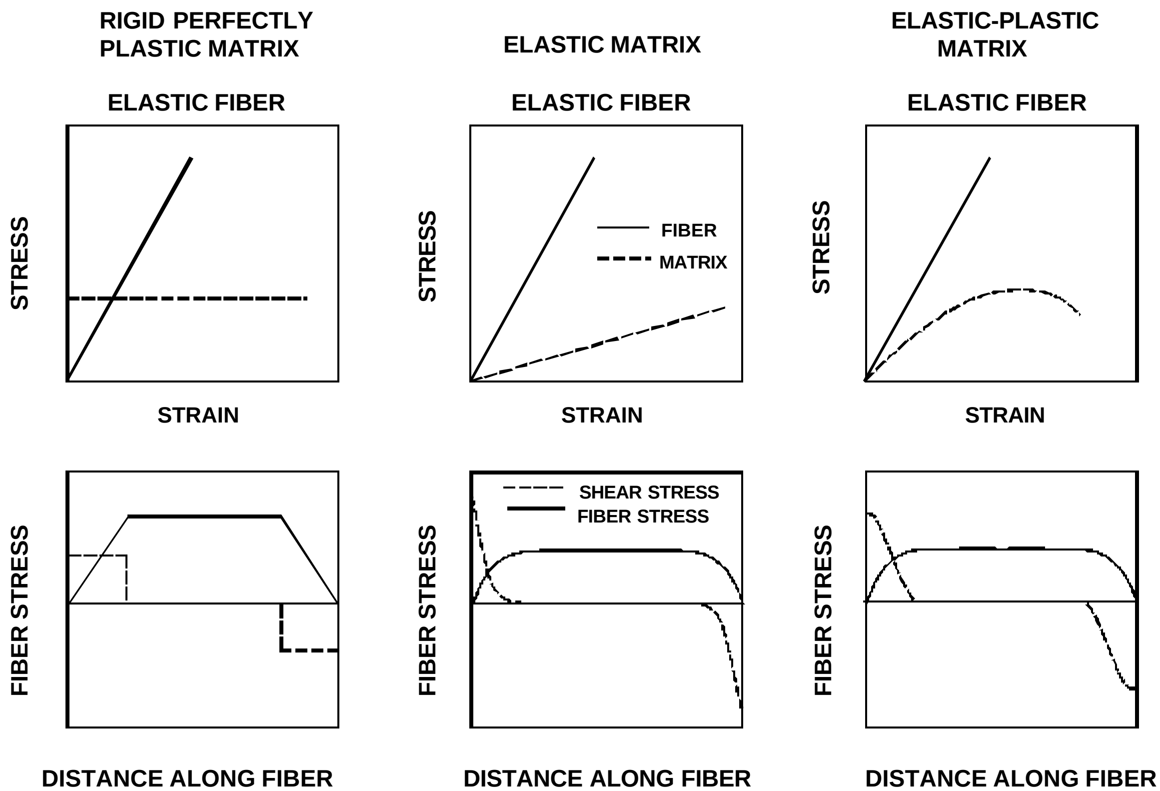

Figure 7-14 Comparison of fiber and shear stress profiles for different matrix behaviors.

Average stress on fiber

For a continuous fiber composite the stress is uniform over the entire length of the fiber, hence the rule-of-mixtures stress is the average stress on the fiber.

$$\bar{\sigma}_f = \frac{E_f}{E_c}\sigma_c \qquad (714)$$

For a discontinuous fiber composite the average stress on the fiber will depend on the length of the fiber and the matrix behavior.

<u>Average stress for elastic fiber-rigid/perfectly plastic matrix</u>

For the case of a rigid/perfectly plastic matrix with an elastic fiber whose length is less than the ineffective length$\left(\ell < \ell_t\right)$ the stress at the center of the fiber is $2\tau_i \ell/d$ while the stress at both ends are zero. The average stress along the entire length of the fiber would be the average of zero and stress at the center, hence

$$\bar{\sigma}_f = \frac{\tau_i \ell}{d} \tag{7.15}$$

For the case of fiber equal to the ineffective length, $\ell = \ell_t$, the stress at the center of the fiber is $E_f \sigma_c / E_c$, hence the average fiber stress is

$$\bar{\sigma}_f = \frac{E_f \sigma_c}{2E_c} \tag{7.16}$$

Fibers that are longer than the ineffective length the average fiber stress can be calculated by

$$\bar{\sigma}_f = \frac{\int_0^\ell \sigma_f \, dx}{\int_0^\ell dx} \tag{7.17}$$

This can be evaluated graphically by measuring the area under the fiber stress-length curve of Fig.7-15 and dividing by the fiber length.

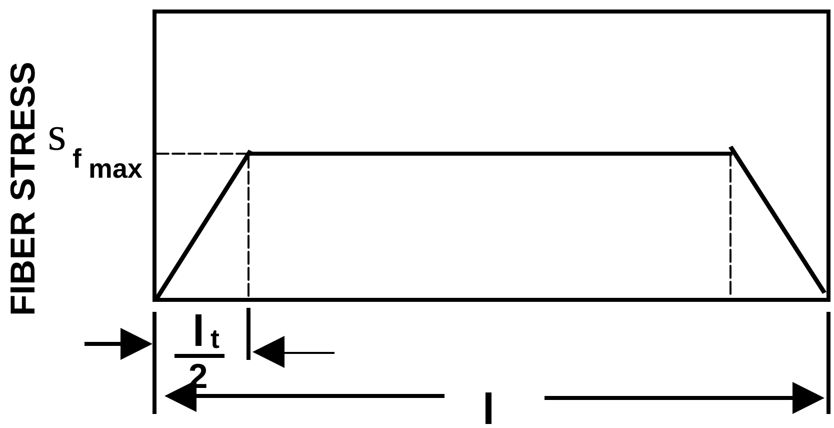

Figure 7-15 Fiber stress distribution for a fiber longer than the ineffective length.

The results of the graphical integration are

$$\bar{\sigma}_f = \sigma_{f\max}\left(1 - \frac{\ell_t}{2\ell}\right) \tag{7.18}$$

If the fiber is very long, $\ell >> \ell_t$, then $\frac{\ell_t}{2\ell}$ approaches zero and Eqn. (7.18) becomes

$$\bar{\sigma}_f = \sigma_{f\max} = \frac{E_f \sigma_c}{E_c} \tag{7.19}$$

<u>Average fiber stress for elastic fiber and elastic matrix</u>

Buy substituting Eqn.(7.12) into the definition of average stress, Eqn. (7.17), the average fiber stress for a discontinuous fiber composite is

$$\bar{s}_f = E_f e_1 \left[1 - \frac{\tanh\left(\frac{n\ell}{r}\right)}{\frac{n\ell}{r}} \right] \tag{7.20}$$

The effect of the elastic behavior of the fibers are often treated in an empirical form as

$$\bar{s}_f = s_{f\max} \left[1 - (1-q)\frac{\ell_t}{\ell} \right] \tag{7.21}$$

where q is the ratio of the area under the ineffective length for the elastic matrix case to the area under the ineffective length for the rigid/perfectly plastic matrix as illustrated in Fig,7-16. The value of q can range from 1 to 0.5 but is generally only slightly greater than 0.5. Thus the error created by the assuming rigid/perfectly plastic matrix is quite small. For longer fibers this difference is negligible. Hence the use of this assumption is quite justified.

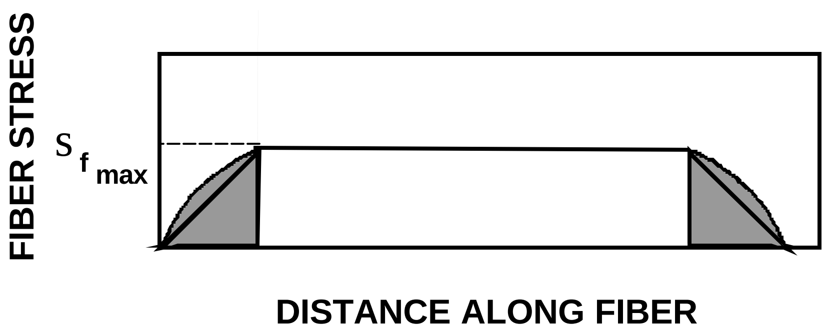

Figure 7-16. Fiber stress plots superimposed for elastic matrix and rigid/perfectly plastic matrix.

Composite stress for discontinuous fiber composites

The stress on a discontinuous fiber composite is given by the rule-of-mixtures based on the average fiber stress.

For $\ell < \ell_t$

$$s_c = \frac{t_i \ell V_f}{d} + s_m (1 - V_f) \tag{7.22}$$

For $\ell = \ell_t$

$$s_c = \frac{1}{2} s_{f\max} V_f + s_m (1 - V_f) \tag{7.23}$$

For $\ell > \ell_t$

$$s_c = s_{f\max} \left(1 - \frac{\ell_t}{2\ell} \right) V_f + s_m (1 - V_f) \tag{7.24}$$

For $\ell >> \ell_t$

$$s_c = s_{f\,\max} V_f + s_m \left(1 - V_f\right) \tag{7.25}$$

Composite strength can be calculated by using the critical length, ℓ_c instead of the ineffective length, ℓ_t and s'_m instead of s_m in Eqns. (7.22), (7.23), (7.24) and (7.25). The rule-of-mixtures composite strength plot for discontinuous fiber composites is then

$$s_{cu} = \bar{s}_{fu} V_f + s'_m \left(1 - V_f\right) \tag{7.26}$$

and

$$s_{cu} = s_{mu} \left(1 - V_f\right) \tag{7.27}$$

depending on whether V_f is above or below V_{crit} respectively. The rule-of-mixtures strength plot for discontinuous fiber composites is shown in Fig.7-17.

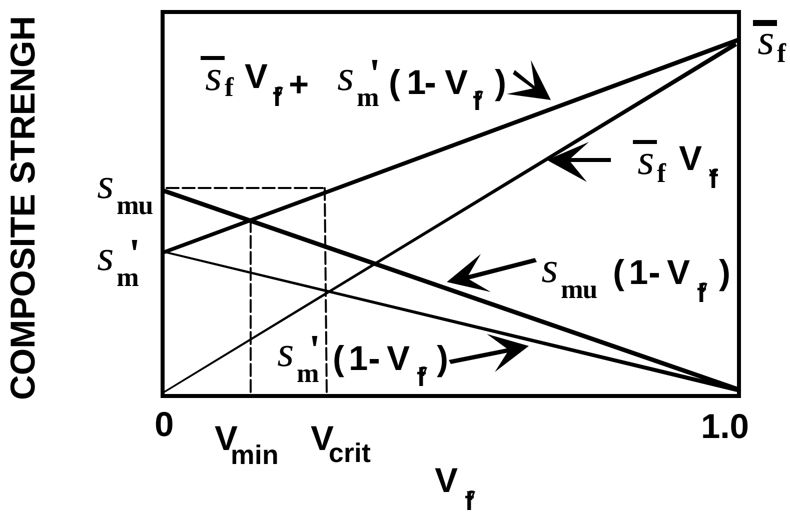

Figure 7-17. Rule-of-mixtures strength for discontinuous fiber composite

Young's modulus of discontinuous fiber composites

Halpin-Tsai equations can be used to calculate the Young's modulus of a discontinuous fiber composite in both the transverse and longitudinal directions. Fig. 7-18 shows the cross section of a round fiber loaded transversely. The shape parameter in the Halpin-Tsai relation, $x = {}^{2a}\!/\!{}_{b} = 2$ for the round fiber. The Young's modulus for the transverse direction in a

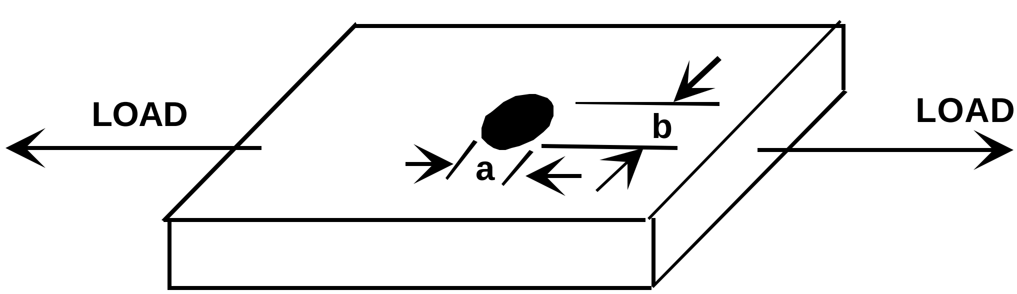

Figure 7-18. Model of a round cross section fiber loaded in the transverse direction
Discontinuous fiber composite is then

$$E_T = E_m \frac{1+2h_T V_f}{1-h_T V_f}$$

where h_T is

$$h_T = \frac{E_f - E_m}{E_f + 2E_m}$$

When a discontinuous fiber is oriented in the load direction as shown in Fig. 7-19 the shape parameter $x_L = 2\left(\ell/d\right)$ where ℓ is the fiber length and d is the fiber diameter.

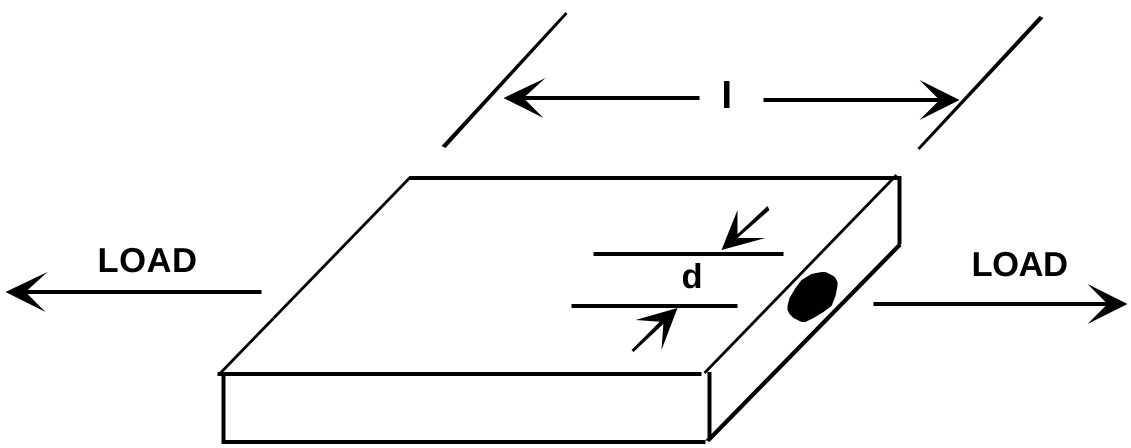

Figure 7-19. Model of discontinuous fiber composite loaded in the longitudinal direction

The Young's modulus in the longitudinal direction is then

$$E_L = E_m \frac{1+\left(2\ell/d\right)h_L V_f}{1-h_L V_f}$$

where

$$h_L = \frac{E_f - E_m}{E_f + \left(2\ell/d\right)E_m}$$

In addition to Young's modulus other properties that can be calculated using the Halpin-Tsai reactions includes shear moduli G_{LT}, G_{TS} and n_{TS}.

The shape parameter, x has been found to be influenced by fiber fraction according to the following empirical relations:

For E_T

$$x_T = 2\frac{a}{b} + 40V_f^{10}$$

For E_L

$$x_L = 2\frac{\ell}{d} + 40V_f^{10}$$

For G_{LT}

$$x_{G_{LT}} = \left(\frac{a}{b}\right)^{1.732} + 40V_f^{10}$$

.

The term $40V_f^{10}$ is relatively small for $V_f < 0.7$ and can be neglected for low volume fractions of reinforcement. Longitudinal and transverse elastic modulus can be used to estimate the Young's modulus and shear modulus of a lamina with randomly oriented reinforcements.

The matrix properties dominate the transverse behavior of composites hence the transverse modulus is given more weight when calculating the Young's modulus for fibers randomly oriented in the plane of the lamina:

$$E_{random} = \frac{5}{8}E_T + \frac{3}{8}E_L$$

An estimate of the shear modulus for the randomly oriented reinforcements can be found from

$$G_{random} = \frac{2}{8}E_T + \frac{1}{8}E_L$$

and for Poisson's ratio

$$n_{random} = \frac{E_{random}}{2G_{random}} - 1$$