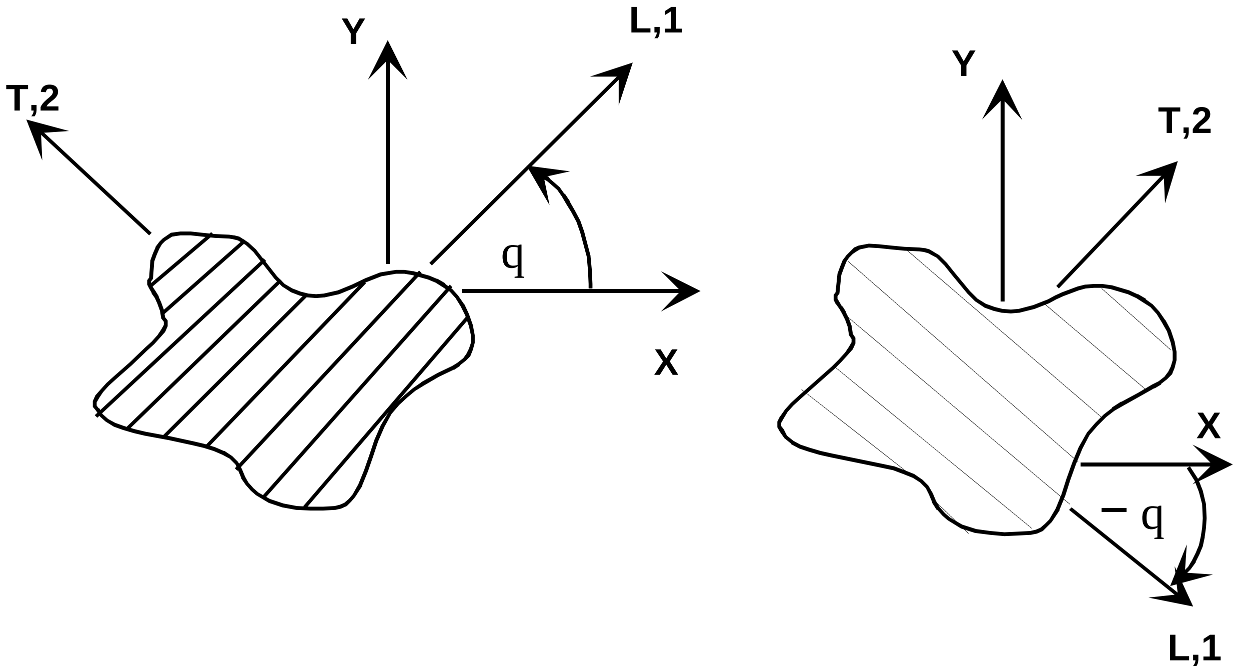
8 Properties of Lamina

ORTHOTROPIC LAMINA

An orthotropic lamina is a sheet with unique and predictable properties and consists of an assemblage of fibers lying in the plane of the sheet and held in place by a matrix. The lamina can be composed of continuous or discontinuous fibers. The labeling conventions used here is shown in Fig. 8-1.



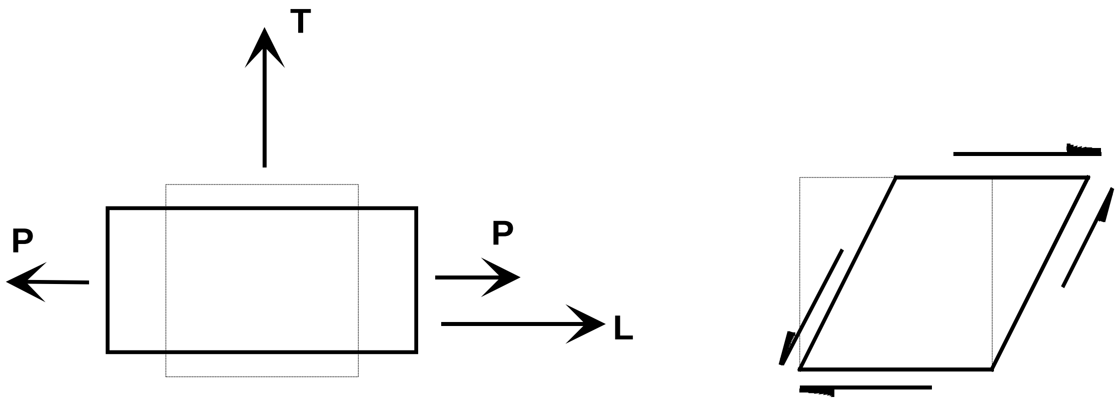
Continuous Fiber Lamina Discontinuous Fiber Lamina

Figure 8-1 Labeling convention for lamina

L and *T* are the principal material directions, also referred to as 1 and 2. The angle formed by the counter clockwise rotation from an arbitrary direction *x* to *L* is +*2*. The clockwise rotation produces -*2*.

Orthotropic material

In an orthotropic material there are two unique mutually orthogonal directions (called the principal material directions) in which tension load causes extension only parallel and perpendicular to the tensile load direction, and shear load causes shear strains only. The effects of extensional and shear loads in the principal directions on the strains in an orthotropic material are shown in Fig.8-3. The unstrained lamina are shown as dotted lines in this figure. The extensional load shown in Fig.8-3a causes only extension in the longitudinal direction and contraction in the transverse direction. Loading in shear is shown in Fig.8-3b which produces only shear strains.

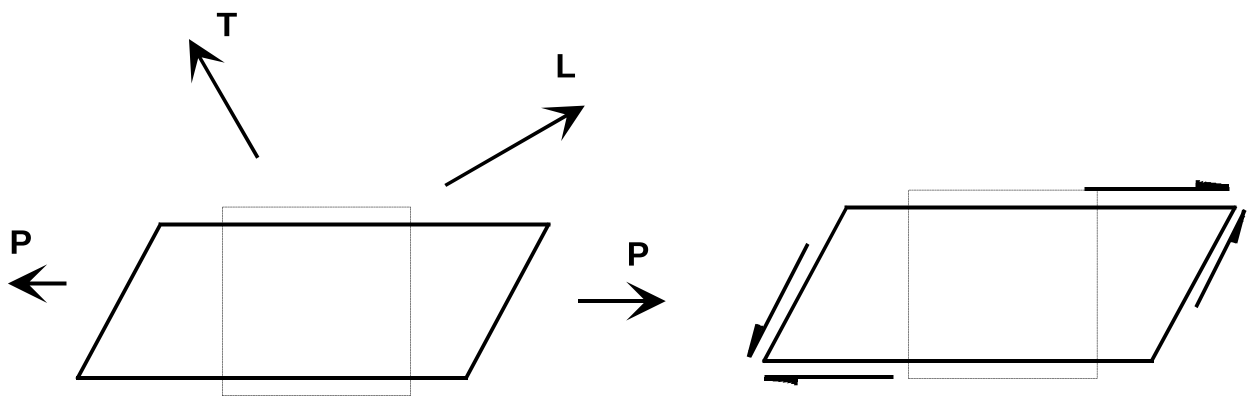


a) Load in principal direction, L

b) Shear load in LT direction

Figure 8-3 Orthotropic lamina loaded in principal directions or isotropic lamina loaded in any direction

Loading, either in shear or in tension in any other direction causes both extensional and shear strains. The effect of load on strain in an orthotropic material loaded in the other than principal directions are shown in Fig.8-4. In an orthotropic material extension and shear are decoupled only in the principal material directions. Fig. 8-4a shows the extensional loads applied in an



a) Extensional load in arbitray direction

b) Shear load in arbitrary direction

Figure 8-4 Orthotropic lamina loaded in an arbitrary direction

arbitrary or non-principal direction producing both extension and shear. In Fig. 8-4b shear loads applied in the non-principal direction produce extensional strains as well as shear strains.

Isotropic material

A tensile load in any direction causes only extension parallel and perpendicular to that direction and shear produces only shear strains. In an isotropic material extension and shear are *always* decoupled and the strains are represented by Fig. 8-3.

Anisotropic material

A tensile load in any direction causes shear as well as parallel and perpendicular extensions. A shear load in any direction causes extensional and shear strains. In anisotropic materials extension and shear are *never* decoupled. Strains are always represented by Fig. 8-4.

HOOKES'S LAW FOR ANISOTROPIC MATERIAL

Hooke’s law can be expressed in full tensor notation by

$$s_{ij} = E_{ijkl} e_{kl}$$

where s_{ij} are the stress components and e_{kl} are the strain components. The elastic constants, E_{ijkl} are fourth ranked tensors. Their subscripts $i,j,k,l = 1,2,3$,hence there are a total of 3^4 or 81 elastic constants. The number of unique constants can be reduced by using symmetry and thermodynamic arguments. Strain symmetry produces the result $E_{ijkl} = E_{ijlk}$ hence there are only 54 unique constants remaining. Stress symmetry produces the result $E_{ijkl} = E_{jikl}$ hence there are now only 36 unique constants remaining. Thermodynamic arguments produce the result $E_{ijkl} = E_{klij}$ hence that leaves only 21 unique elastic constants as follows,

$$\begin{bmatrix} E_{1111} & E_{1122} & E_{1133} & E_{1123} & E_{1131} & E_{1112} \\ E_{1122} & E_{2222} & E_{2233} & E_{2223} & E_{2231} & E_{2212} \\ E_{1133} & E_{2233} & E_{3333} & E_{3323} & E_{3331} & E_{3312} \\ E_{1123} & E_{2223} & E_{3323} & E_{2323} & E_{2331} & E_{2312} \\ E_{1131} & E_{2231} & E_{3331} & E_{2331} & E_{3131} & E_{3112} \\ E_{1112} & E_{2212} & E_{3312} & E_{2312} & E_{3112} & E_{1212} \end{bmatrix}$$

Hooke's law can also be written in contracted notation which replace paired subscripts with a single subscript according to the following: 1 for 11, 2 for 22, 3 for 33, 4 for 23, 5 for 31, and 6 for 12.

$$s_i = C_{ij} e_j$$

where subscripts i, j can have the values 1,2,3,4,5, and 6.

TRANSFORMATION OF ELASTIC CONSTANTS

To transform an elastic constant from the X axis to the X' axis use the following:

$$E'_{mnrs} = a_{im} a_{jn} a_{kr} a_{ls} E_{ijkl} \qquad (8.1)$$

where a_{im}, a_{jn}, a_{kr}, a_{ls} are direction cosines.

For one plane of symmetry, say X_1X_2 as illustrated in Fig.8-5 then $X_1 = X'_1$, $X_2 = X'_2$, and $X_3 = -X'_3$

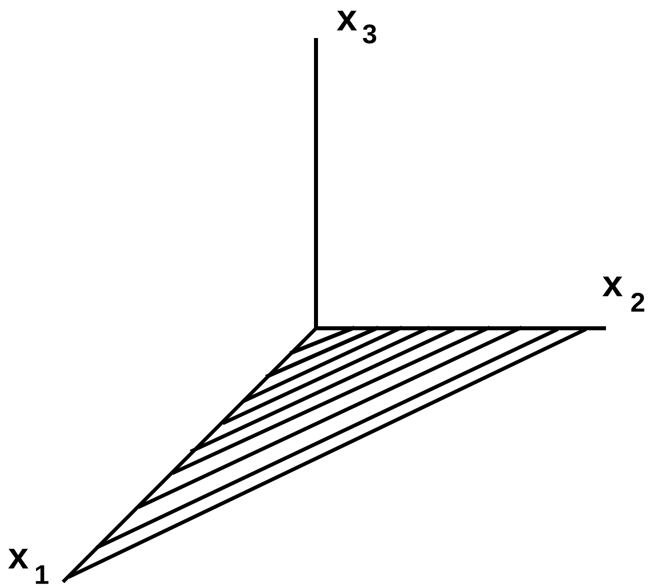


Figure 8-5 Plane of symmetry X_1X_2 in an orthotropic material

The direction cosines for transformation through this plane of symmetry are:

	X'_1	X'_2	X'_3
X_1	$a_{11} = 1$	$a_{12} = 0$	$a_{13} = 0$
X_2	$a_{21} = 0$	$a_{22} = 1$	$a_{23} = 0$
X_3	$a_{31} = 0$	$a_{32} = 0$	$a_{33} = -1$

Using the above determined direction cosines in the transformation Eqn.(8.1) produces the following result

$$E'_{1111} = E_{1111}$$
$$E'_{1122} = E_{1122}$$
$$E'_{1131} = -E_{1131}$$

The result $E'_{1131} = -E_{1131}$ can only be true if $E_{1131} = 0$. Examination of the stiffness matrix indicates that odd multiples of a_{33} will give negative and hence zero values for E_{ijkl}. Applying this observation to all the coefficients then the following E_{ijkl} are zero:

$$E_{1113}, E_{1213}, E_{2223}, E_{1223}, E_{1123}, E_{1333}, E_{2213}, E_{2333}$$

The same technique can be applied to a second plane, say X_2X_3

	X'_1	X'_2	X'_3
X_1	$a_{11} = -1$	$a_{12} = 0$	$a_{13} = 0$
X_2	$a_{21} = 0$	$a_{22} = 1$	$a_{23} = 0$
X_3	$a_{31} = 0$	$a_{32} = 0$	$a_{33} = 1$

This time odd multiples of a_{11} give negative (hence zero) E_{ijkl}

T		**T**		**T**	**T**	
E_{1112}	E_{2231}	E_{2212}	E_{3331}	E_{3312}	E_{2331}	E_{2312}

The **T**'ed ones are new *E*'s eliminated by the second plane of symmetry, unchecked *E*'s fit the criteria but were already eliminated by the first plane of symmetry. Applying the third plane of symmetry does not eliminate any other coefficients.

Hooke's Law for the orthotropic material is

$$\begin{Bmatrix} s_1 \\ s_2 \\ s_3 \\ t_{23} \\ t_{31} \\ t_{12} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} \begin{Bmatrix} e_1 \\ e_2 \\ e_3 \\ g_{23} \\ g_{31} \\ g_{12} \end{Bmatrix} \tag{8.2}$$

<u>REDUCED STIFFNESS MATRIX FOR PLANE STRESS CONDITION</u>

For a two dimensional lamina (sheet) only the stresses in the plane of the lamina are non-zero. Hence the streses through the thickness are zero under plane stress

$$s_3 = t_{23} = t_{31} = 0$$

Applying these values to Eqn.(8.2)

$$\begin{Bmatrix} s_1 \\ s_2 \\ 0 \\ 0 \\ 0 \\ t_{12} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} \begin{Bmatrix} e_1 \\ e_2 \\ e_3 \\ g_{23} \\ g_{31} \\ g_{12} \end{Bmatrix} \tag{8.3}$$

then Hooke's Law (Eqn.(8.3)) for plane stress gives

$$\begin{aligned} s_1 &= C_{11}e_1 + C_{12}e_2 + C_{13}e_3 \\ s_2 &= C_{12}e_1 + C_{22}e_2 + C_{23}e_3 \\ 0 &= C_{13}e_1 + C_{23}e_2 + C_{33}e_3 \\ t_{12} &= C_{66}g_{12} \end{aligned} \tag{8.4}$$

The strain, e_3 can be expressed in terms of e_1 and e_2

$$e_3 = -\frac{C_{13}e_1}{C_{33}} - \frac{C_{23}e_2}{C_{33}} \tag{8.5}$$

The Eqns. (8.4) can now be written

$$\begin{aligned} s_1 &= \left(C_{11} - \frac{C_{13}^2}{C_{33}} \right) e_1 + \left(C_{12} - \frac{C_{13}C_{23}}{C_{33}} \right) e_2 \\ s_2 &= \left(C_{12} - \frac{C_{13}C_{23}}{C_{33}} \right) e_1 + \left(C_{22} - \frac{C_{23}^2}{C_{33}} \right) e_2 \\ t_{12} &= C_{66}g_{12} \end{aligned} \tag{8.6}$$

Using the following notations:

$$Q_{11} = C_{11} - \frac{C_{13}^2}{C_{33}},\ Q_{12} = C_{12} - \frac{C_{13}C_{23}}{C_{33}},\ Q_{22} = C_{22} - \frac{C_{23}^2}{C_{33}} \text{ and } Q_{66} = C_{66} \tag{8.7}$$

Then Hooke's law for plane stress condition takes the reduced form

$$\begin{Bmatrix} s_1 \\ s_2 \\ t_{12} \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \begin{Bmatrix} e_1 \\ e_2 \\ g_{12} \end{Bmatrix} \tag{8.8}$$

For lamina that are transversely isotropic, which is generally the case for most practical composite plies, $C_{13} = C_{12}$ and $C_{33} = C_{22}$.

Hooke's law for plane stress in terms of compliance is

$$\begin{Bmatrix} e_1 \\ e_2 \\ g_{12} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & 0 \\ S_{12} & S_{22} & 0 \\ 0 & 0 & S_{66} \end{bmatrix} \begin{Bmatrix} s_1 \\ s_2 \\ t_{12} \end{Bmatrix} \tag{8.9}$$

S_{ij}'s and Q_{ij}'s are mutually inverse

$$[I]=[S][Q]$$

Inverting $[S]$

$$Q_{11}=\frac{S_{22}}{S_{11}S_{22}-S_{12}^2}$$
$$Q_{22}=\frac{S_{11}}{S_{11}S_{22}-S_{12}^2} \qquad (8.10)$$
$$Q_{12}=-\frac{S_{12}}{S_{11}S_{22}-S_{12}^2}$$
$$Q_{66}=\frac{1}{S_{66}}$$

Inverting $[Q]$

$$S_{11}=\frac{Q_{22}}{Q_{11}Q_{22}-Q_{12}^2}$$
$$S_{22}=\frac{Q_{11}}{Q_{11}Q_{22}-Q_{12}^2} \qquad (8.11)$$
$$S_{12}=-\frac{Q_{12}}{Q_{11}Q_{22}-Q_{12}^2}$$
$$S_{66}=\frac{1}{Q_{66}}$$

Compliance and stiffness in terms of engineering constants

Using Eqn. (8.9) and the definitions of Young’s ratio and Poisson’s ratio the values of the compliance constants can be found to be

$$S_{11}=\frac{1}{E_1}$$
$$S_{22}=\frac{1}{E_2}$$
$$S_{12}=-\frac{n_{12}}{E_1}=-\frac{n_{21}}{E_2} \qquad (8.12)$$
$$S_{66}=\frac{1}{G_{12}}$$

Substituting Eqns. (8.12) into Eqns. (8.10) gives

$$Q_{11}=\frac{E_1}{1-n_{12}n_{21}}$$
$$Q_{22}=\frac{E_2}{1-n_{12}n_{21}} \qquad (8.13)$$
$$Q_{12}=\frac{n_{21}E_1}{1-n_{12}n_{21}}=\frac{n_{12}E_2}{1-n_{12}n_{21}}$$
$$Q_{66}=G_{12}$$

From Eqns. (8.12) it is also clear that

$$\frac{n_{12}}{n_{21}} = \frac{E_1}{E_2} \tag{8.14}$$

STRESS-STRAIN RELATION FOR OFF-AXIS LAMINA

Hooke's Law is readily expressed by Eqns. (8.8) and (8.9) for stresses and strains in the principal material direction of a composite. In real composite lamina stresses are often applied in arbitrary direction *x* at an angle q from the principal directions as seen in Fig.8-6.

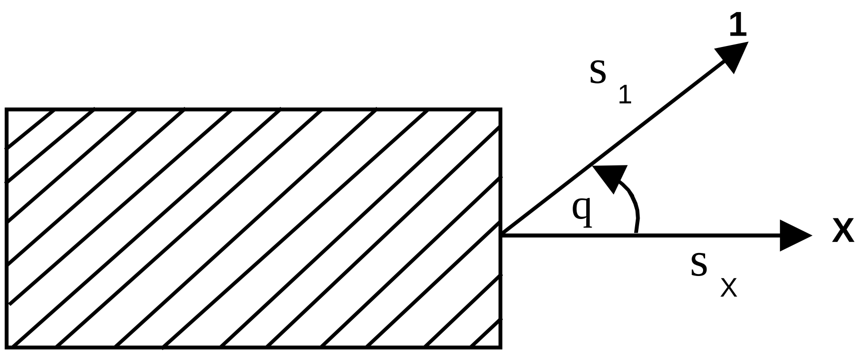


Figure 8-6 Stress applied to a composite lamina in an arbitrary direction

Transforming stress F_x in the arbitrary direction *x* to the F_1 in the principal material direction *1*

$$\begin{Bmatrix} s_1 \\ s_2 \\ t_{12} \end{Bmatrix} = [T] \begin{Bmatrix} s_x \\ s_y \\ t_{xy} \end{Bmatrix} \tag{8.15}$$

Transforming strain from the arbitrary to the principle direction is performed by

$$\begin{Bmatrix} e_1 \\ e_2 \\ \frac{1}{2} g_{12} \end{Bmatrix} = [T] \begin{Bmatrix} e_x \\ e_y \\ \frac{1}{2} g_{xy} \end{Bmatrix} \tag{8.16}$$

where g_{xy} is engineering strain, $\frac{1}{2} g_{xy}$ is tensorial strain and [*T*] is second order transformation matrix given by

$$[T]=\begin{bmatrix} \cos^2\theta & \sin^2\theta & 2\cos\theta\sin\theta \\ \sin^2\theta & \cos^2\theta & -2\cos\theta\sin\theta \\ -\cos\theta\sin\theta & \cos\theta\sin\theta & \left(\cos^2\theta-\sin^2\theta\right) \end{bmatrix} \tag{8.17}$$

To transform from principal direction to arbitrary direction, then $2 = -2$ and transformation matrix becomes

$$[T]^{-1}=\begin{bmatrix} \cos^2\theta & \sin^2\theta & -2\cos\theta\sin\theta \\ \sin^2\theta & \cos^2\theta & 2\cos\theta\sin\theta \\ \cos\theta\sin\theta & -\cos\theta\sin\theta & \left(\cos^2\theta-\sin^2\theta\right) \end{bmatrix} \tag{8.18}$$

Hence, Eqn. (8.16) is the inverse of Eqn.(8.17).

Transforming stress from principal to arbitrary direction *x*

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = [T]^{-1} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix} \tag{8.19}$$

Transforming strain from principal to arbitrary direction *x*

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \frac{1}{2}\gamma_{xy} \end{Bmatrix} = [T]^{-1} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \frac{1}{2}\gamma_{12} \end{Bmatrix} \tag{8.20}$$

HOOKES LAW FOR STRESSES AND STRAINS APPLIED IN ARBITRARY DIRECTION

To obtain Hooke's Law for loading in a direction other than the principal material directions the stiffness or compliance coefficients which are defined in the principal directions have to be transformed to the arbitrary direction of loading. The sequence of operations to perform this transformation on the stiffness coefficients, given the engineering strain in the arbitrary direction, is as follows:

(1) Convert engineering strain in the arbitrary direction to tensorial strain in the arbitrary direction.

(2) Transform tensorial strain in the arbitrary direction to tensorial strain in the principal material direction.

(3) Convert tensorial strain in the principal material direction back to the engineering strain in the principal material direction.

(4) Find engineering strain in the principal material direction to stress in the principal material direction using Hooke's Law for orthotropic material.

(5) Transform stress in the principal material direction to stress in the original arbitrary direction.

A convenient way to perform step (1) is to multiply the engineering strain by the inverse of the Reuter's matrix.

$$\begin{Bmatrix} e_x \\ e_y \\ \frac{1}{2} g_{xy} \end{Bmatrix} = [R]^{-1} \begin{Bmatrix} e_x \\ e_y \\ g_{xy} \end{Bmatrix} \qquad (8.21)$$

where

$$[R]^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}$$

In step (2) apply the tensorial strain transformation of Eqn. (8.16) to find the tensorial strains in the principal material direction.

$$\begin{Bmatrix} e_1 \\ e_2 \\ \frac{1}{2} g_{12} \end{Bmatrix} = [T] \begin{Bmatrix} e_x \\ e_y \\ \frac{1}{2} g_{xy} \end{Bmatrix} \qquad (8.22)$$

The stiffness matrix $[Q]$ is defined in terms of the engineering strains in the principal material direction, hence in step (3) the tensorial strains in the principal material direction is converted to engineering strain. This can be conveniently performed using the Reuter's matrix

$$\begin{Bmatrix} e_1 \\ e_2 \\ g_{12} \end{Bmatrix} = [R] \begin{Bmatrix} e_1 \\ e_2 \\ 1/2\, g_{12} \end{Bmatrix} \qquad (8.23)$$

where

$$[R] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Now step (4) can be performed using Eqn.(8.8) to find the stresses in the principal material directions

$$\begin{Bmatrix} s_1 \\ s_2 \\ t_{12} \end{Bmatrix} = [Q] \begin{Bmatrix} e_1 \\ e_2 \\ g_{12} \end{Bmatrix} \tag{8.24}$$

The stress in the principal material directions fron Eqn (8.19) can be transformed back to the arbitrary direction by in inverse transformation matrix

$$\begin{Bmatrix} s_x \\ s_y \\ t_{xy} \end{Bmatrix} = [T]^{-1} \begin{Bmatrix} s_1 \\ s_2 \\ t_{12} \end{Bmatrix} \tag{8.25}$$

Showing all of these operations in one equation gives

$$\begin{Bmatrix} s_x \\ s_y \\ t_{xy} \end{Bmatrix} = [T]^{-1}[Q][R][T][R]^{-1} \begin{Bmatrix} e_x \\ e_y \\ g_{xy} \end{Bmatrix} \tag{8.26}$$

Premultiplying the conversion, transformation and stiffness matrices gives the *transformed stiffness matrix*

$$[\bar{Q}] = [T]^{-1}[Q][R][T][R]^{-1} \tag{8.27}$$

It can be shown that Eqn. (8.27) can be reduced to

$$[\bar{Q}] = [T]^{-1}[Q][T]^{-T} \tag{8.28}$$

The transformed stiffness matrix expressed in terms of the individual coefficients is

$$[\bar{Q}] = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix} \tag{8.29}$$

where

$$\begin{aligned}
\bar{Q}_{11} &= Q_{11}\cos^4 q + 2\left(Q_{12}+2Q_{66}\right)\sin^2 q\cos^2 q + Q_{22}\sin^4 q \\
\bar{Q}_{22} &= Q_{11}\sin^4 q + 2\left(Q_{12}+2Q_{66}\right)\sin^2 q\cos^2 q + Q_{22}\cos^4 q \\
\bar{Q}_{12} &= \left(Q_{11}+Q_{22}-4Q_{66}\right)\sin^2 q\cos^2 q + Q_{12}\left(\sin^4 q+\cos^4 q\right) \\
\bar{Q}_{16} &= \left(Q_{11}-Q_{12}-2Q_{66}\right)\sin q\cos^3 q + \left(Q_{12}-Q_{22}+2Q_{66}\right)\sin^3 q\cos q \\
\bar{Q}_{26} &= \left(Q_{11}-Q_{12}-2Q_{66}\right)\sin q^3\cos q + \left(Q_{12}-Q_{22}+2Q_{66}\right)\sin q\cos^3 q \\
\bar{Q}_{66} &= \left(Q_{11}+Q_{22}-2Q_{12}-2Q_{66}\right)\sin^2 q\cos^2 q + Q_{66}\left(\sin^4 q+\cos^4 q\right)
\end{aligned} \tag{8.30}$$

By a similar process the tansformed compliance matrix can be found for finding the stains in an arbitary direction form the stresses in an arbitrary direction

$$\begin{Bmatrix} e_x \\ e_y \\ g_{xy} \end{Bmatrix} = [R][T]^{-1}[R]^{-1}[S][T]\begin{Bmatrix} s_x \\ s_y \\ t_{xy} \end{Bmatrix} \tag{8.31}$$

where

$$\left[\bar{S}\right] = [R][T]^{-1}[R]^{-1}[S][T] \tag{8.32}$$

or

$$\left[\bar{S}\right] = [T]^T[S][T] \tag{8.33}$$

The *transformed compliance matrix* expressed in terms of the individual coefficients is

$$\left[\bar{S}\right] = \begin{bmatrix} \bar{S}_{11} & \bar{S}_{12} & \bar{S}_{16} \\ \bar{S}_{12} & \bar{S}_{22} & \bar{S}_{26} \\ \bar{S}_{16} & \bar{S}_{26} & \bar{S}_{66} \end{bmatrix} \tag{8.34}$$

where

$$\begin{aligned}
\bar{S}_{11} &= S_{11}\cos^4 q + S_{22}\sin^4 q + \left(2S_{12}+S_{66}\right)\sin^2 q\cos^2 q \\
\bar{S}_{22} &= S_{11}\sin^4 q + S_{22}\cos^4 q + \left(2S_{12}+S_{66}\right)\sin^2 q\cos^2 q \\
\bar{S}_{12} &= \left(S_{11}+S_{22}-S_{66}\right)\sin^2 q\cos^2 q + S_{12}\left(\cos^4 q+\sin^4 q\right) \\
\bar{S}_{16} &= \left(2S_{11}-2S_{12}-S_{66}\right)\sin q\cos^3 q - \left(2S_{22}-2S_{12}-S_{66}\right)\sin^3 q\cos q \\
\bar{S}_{26} &= \left(2S_{11}-2S_{12}-S_{66}\right)\sin^3 q\cos q - \left(2S_{22}-2S_{12}-S_{66}\right)\sin q\cos^3 q \\
\bar{S}_{66} &= 2\left(2S_{11}+2S_{22}-4S_{12}-S_{66}\right)\sin^2 q\cos^2 q + S_{66}\left(\cos^4 q+\sin^4 q\right)
\end{aligned} \tag{8.35}$$

INVARIENT FORMS OF STIFFNESS COEFFICIENTS

The reduced transformed stiffness coefficients can be expressed in terms of five invariant constants that depend upon the untransformed, (i.e. independent of direction) stiffness coefficients. These invariant coefficients are

$$
\begin{aligned}
U_1 &= \frac{3Q_{11} + 3Q_{22} + 2Q_{12} + 4Q_{66}}{8} \\
U_2 &= \frac{Q_{11} - Q_{22}}{2} \\
U_3 &= \frac{Q_{11} + Q_{22} - 2Q_{12} - 4Q_{66}}{8} \\
U_4 &= \frac{Q_{11} + Q_{22} + 6Q_{12} - 4Q_{66}}{8} \\
U_5 &= \frac{Q_{11} + Q_{22} - 2Q_{12} + 4Q_{66}}{8}
\end{aligned}
\qquad (8.36)
$$

Using these invariants coefficient the transformed stiffness coefficients can be written as

$$
\begin{aligned}
\bar{Q}_{11} &= U_1 + U_2 \cos 2q + U_3 \cos 4q \\
\bar{Q}_{22} &= U_1 - U_2 \cos 2q + U_3 \cos 4q \\
\bar{Q}_{12} &= U_4 - U_3 \cos 4q \\
\bar{Q}_{16} &= -\frac{1}{2} U_2 \sin 2q - U_3 \sin 4q \\
\bar{Q}_{26} &= -\frac{1}{2} U_2 \sin 2q + U_3 \sin 4q \\
\bar{Q}_{66} &= U_5 - U_3 \cos 4q
\end{aligned}
\qquad (8.37)
$$

Each of the terms on the right hand side of Eqn.(8.33) can be plotted individually and then summed to produce the transformed stiffness coefficient. This summation is shown schematically in Fig.8-7.

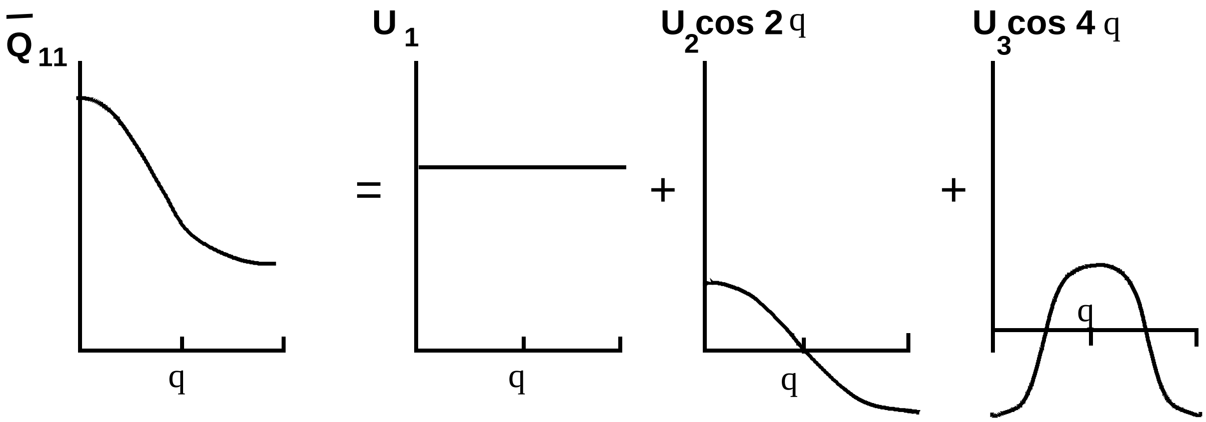


Figure 8-7. Transformed reduced stiffness coefficient constructed form invariant coefficients

In this example the coefficient $\bar{Q}_{11}$ consists of the summation of a constant, cosine term and a havercosine term.

TRANSFORMATION OF ENGINEERING CONSTANTS

From measured elastic engineering constants in the principal material directions, shown in Fig.8-8, the elastic constants in any arbitrary direction can be determined.

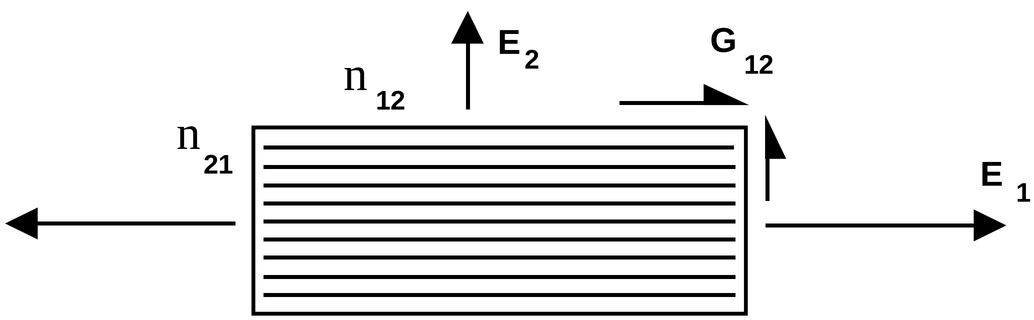


Figure 8-8. Elastic engineering constants in the principal material directions

This can be performed as follows:

Consider the following elastic strains in the principal material direction: e_1, e_2 and g_{12}. Transform these strains to the arbitrary direction, let us say at some angle, q counterclockwise from the principal direction, 1.

$$
\begin{aligned}
e_x &= e_1 \cos^2 q + e_2 \sin^2 q - g_{12} \cos q \sin q \\
e_y &= e_1 \sin^2 q + e_2 \cos^2 q + g_{12} \cos q \sin q \\
g_{xy} &= 2e_1 \cos q \sin q - 2e_2 \cos q \sin q + g_{12} \left(\cos^2 q - \sin^2 q\right)
\end{aligned}
\tag{8.38}
$$

Using Hooke's Law the strains in the principal material directions can be expressed in terms of engineering elastic constants and stresses in the principal material directions.

$$\begin{aligned} \varepsilon_1 &= \frac{\sigma_1}{E_1} - \frac{\nu_{12}\sigma_2}{E_1} \\ \varepsilon_2 &= \frac{\sigma_2}{E_2} - \frac{\nu_{21}\sigma_1}{E_2} \\ \gamma_{12} &= \frac{\tau_{12}}{G_{12}} \end{aligned} \tag{8.39}$$

If we apply σ_x as the only non-zero arbitrary stress then the stresses in the principal directions are

$$\begin{aligned} \sigma_1 &= \sigma_x \cos^2\theta \\ \sigma_2 &= \sigma_x \sin^2\theta \\ \tau_{12} &= -\sigma_x \cos\theta \sin\theta \end{aligned} \tag{8.40}$$

Combining Eqns.(8.39) and (8.40) with (8.38) yields the strains in arbitrary direction in terms of the engineering constants for the principal material direction,

$$\varepsilon_x = \sigma_x \left[\frac{\cos^4\theta}{E_1} + \frac{\sin^4\theta}{E_2} + \frac{1}{4}\left(\frac{1}{G_{12}} - \frac{2\nu_{12}}{E_1} \right) \sin^2 2\theta \right] \tag{8.41a}$$

$$\varepsilon_y = -\sigma_x \left[\frac{\nu_{12}}{E_1} - \frac{1}{4}\left(\frac{1}{E_1} + \frac{2\nu_{12}}{E_1} + \frac{1}{E_2} - \frac{1}{G_{12}} \right) \sin^2 2\theta \right] \tag{8.41b}$$

$$\gamma_{xy} = -\sigma_x \sin 2\theta \left[\frac{\nu_{12}}{E_1} + \frac{1}{E_2} - \frac{1}{2G_{12}} - \cos^2\theta \left(\frac{1}{E_1} + \frac{2\nu_{12}}{E_1} + \frac{1}{E_2} - \frac{1}{G_{12}} \right) \right] \tag{8.41c}$$

The Young's modulus in the arbitrary direction can now be found as follows:

From Eqn, (8.41a)

$$E_x = \frac{\sigma_x}{\varepsilon_x} = \frac{1}{\left[\frac{\cos^4\theta}{E_1} + \frac{\sin^4\theta}{E_2} + \frac{1}{4}\left(\frac{1}{G_{12}} - \frac{2\nu_{12}}{E_1} \right) \sin^2 2\theta \right]} \tag{8.42}$$

If Eqn. (8.41a) is evaluated at $\theta + 90$

$$E_y = \frac{\sigma_y}{\varepsilon_y} = \frac{1}{\left[\frac{\cos^4\theta}{E_2} + \frac{\sin^4\theta}{E_1} + \frac{1}{4}\left(\frac{1}{G_{12}} - \frac{2\nu_{12}}{E_1} \right) \sin^2 2\theta \right]} \tag{8.43}$$

If we apply t_{xy} as the only non-zero stress in the above described procedure the shear modulus in the arbitrary direction is

$$G_{xy} = \frac{1}{2\left(\frac{2}{E_1} + \frac{2}{E_2} + \frac{4n_{12}}{E_1} - \frac{1}{G_{12}}\right)\sin^2 2q + \frac{1}{G_{12}}\left(\sin^4 q + \cos^4 q\right)} \tag{8.44}$$

Using the definition of Poisson's ratio and Eqns. (8.41a) and (8.41b)

$$n_{xy} = -\frac{e_y}{e_x} = E_x\left[\frac{n_{12}}{E_1}\left(\sin^4 q - \cos^4 q\right) - \left(\frac{1}{E_1} + \frac{1}{E_2} - \frac{1}{G_{12}}\right)\sin^2 2q\right] \tag{8.45}$$

The stress s_x can also produce shear strain g_{xy}, and shear stress t_{xy} can produce extensional strain e_x, these can be related though a the cross coefficient m_x by

$$g_{xy} = -\frac{m_x s_x}{E_1} \tag{8.46a}$$

and

$$e_x = -\frac{m_x t_{xy}}{E_1} \tag{8.46b}$$

Likewise for stress s_y and strain g_{xy}, and stress t_{xy} and strain e_y, using the cross coefficient m_y,

$$g_{xy} = -\frac{m_y s_y}{E_1} \tag{8.46a}$$

and

$$e_y = -\frac{m_y t_{xy}}{E_1} \tag{8.47b}$$

The cross coefficient m_x can be found from Eqn. (8.4c) as

$$m_x = \sin 2q\left[n_{12} + \frac{E_1}{E_2} - \frac{E_1}{2G_{12}} - \cos^2 q\left(1 + 2n_{12} + \frac{E_1}{E_2} - \frac{E_1}{G_{12}}\right)\right] \tag{8.48}$$

The cross coefficient m_y can be similarly found

$$m_y = \sin 2q \left[n_{12} + \frac{E_1}{E_2} - \frac{E_1}{2G_{12}} - \sin^2 q \left(1 + 2n_{12} + \frac{E_1}{E_2} - \frac{E_1}{G_{12}} \right) \right] \tag{8.49}$$

These transformed engineering elastic constants can be used in the compliance matrix for stresses and strains in the arbitrary directions (i.e. off-axis loading)

$$\begin{Bmatrix} e_x \\ e_y \\ g_{xy} \end{Bmatrix} = \begin{bmatrix} \frac{1}{E_x} & -\frac{n_{xy}}{E_x} & -\frac{m_x}{E_1} \\ -\frac{n_{xy}}{E_x} & \frac{1}{E_y} & -\frac{m_y}{E_1} \\ -\frac{m_x}{E_1} & -\frac{m_y}{E_1} & \frac{1}{G_{xy}} \end{bmatrix} \begin{Bmatrix} s_x \\ s_y \\ t_{xy} \end{Bmatrix} \tag{8.50}$$

The off-axis Young's modulus Eqns. (8.42 and (8.43) and shear modulus Eqn. (8.44) for a typical carbon fiber/polymer matrix composite are plotted over the range of orientations in Fig. 8-9

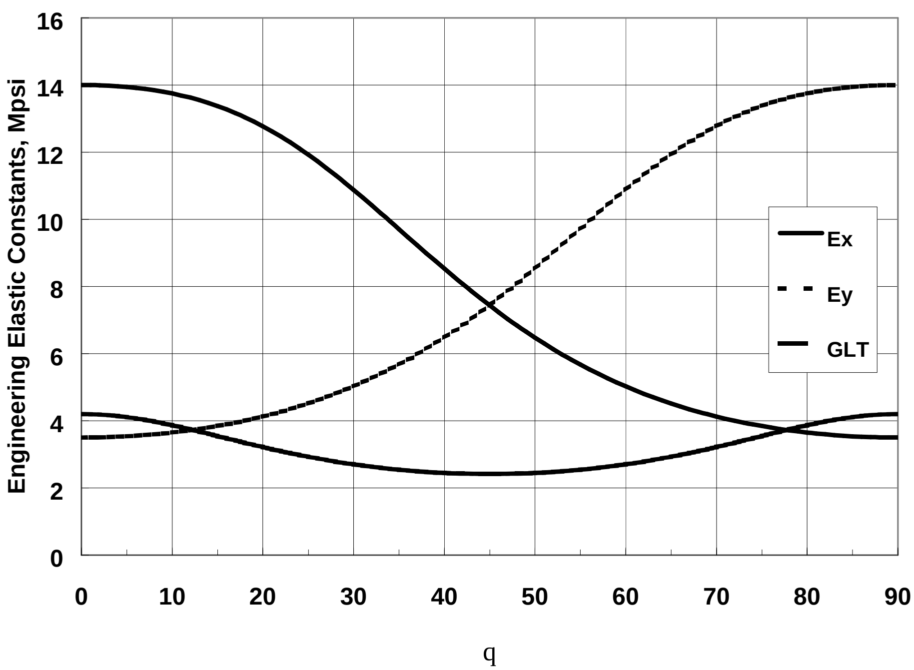


Figure 8-9 Off-axis Young's modulus and shear modulus for typical carbon/epoxy lamina

Fig. 8-10 is a plot of the off-axis Poisson's ratio Eqn.(8.45) and off-axis cross coefficients over the range of orientations Eqns. (8.46) and (8.47) for a typical carbon fiber/polymer matrix lamina.

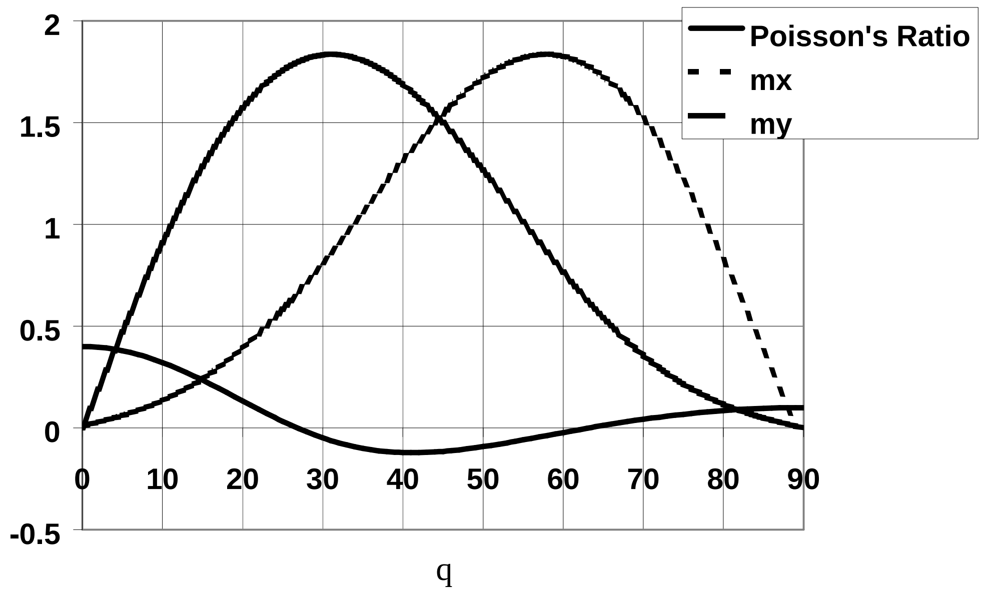


Figure 8-10 Off-axis Poisson’s ratio and cross coefficients for typical carbon fiber/polymer matrix lamina.

RESTRICTIONS ON ELASTIC CONSTANTS

Isotropic materials

Thermodynamic considerations require that Young’s modulus and shear modulus are positive values. The work of deformation, se and tg are positive then

$$\frac{s^2}{E} > 0$$

and

$$\frac{t^2}{G} > 0$$

For isotropic materials the relation between the Young’s modulus and shear modulus is

$$G = \frac{E}{2(1+n)}$$

For G and E to be positive $n > -1$.

The volumetric strain, f resulting from hydrostatic pressure P is

$$f = e_x + e_y + e_z = \frac{P}{K}$$

where K the bulk modulus is

$$K = \frac{E}{3(1-2n)}$$

For K and E to be positive $n < \frac{1}{2}$

For all the elastic moduli to be positive then

$$-1 < n < \frac{1}{2}$$

Orthotropic materials

The relation between the elastic constants are more complex for orthotropic materials. Applying thermodynamic restraints to

$$C_{11}, C_{22}, C_{33}, C_{44}, C_{55}, C_{66}, > 0$$

For the plane stress condition Q_{11}, Q_{22}, $Q_{66} > 0$, then for example

$$Q_{11} = \frac{E_1}{1 - n_{12} n_{21}}$$

and therefore $1 - n_{12} n_{21} > 0$. This is true if $\sqrt{n_{12} n_{21}} < 1$. From this it follows that

$$\sqrt{\frac{E_1}{E_2}} > |n_{12}|$$

and

$$\sqrt{\frac{E_2}{E_1}} > |n_{21}|$$

It is often prudent to verify that the elastic properties used to calculate compliance and stiffness coefficients do not violate these criteria.