Capillary Phenomena in Assemblies of Parallel Cylinders

II. Capillary Rise in Systems with More Than Two Cylinders

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Received January 24, 1969

A previously developed theory of the capillary rise between two closely spaced cylinders has been extended to systems of three and four equidistant cylinders as a function of the distance of separation and the contact angle. The multicylinder system has also been treated.

At small separations between the cylinders there is a high saddle-shaped meniscus between each pair of adjacent cylinders and a low meniscus in the channel between the three or four cylinders. Above a certain distance of separation one meniscus remains.

The analysis used to solve these problems can be utilized also to obtain the meniscus height in closed capillaries of noncircular cross section. As examples, the cases of triangular and square capillaries are discussed.

In all systems it is assumed that the height of the meniscus is much greater than the radius of the cylinders (or the dimensions of the closed capillaries). For some cases the errors involved are estimated.

INTRODUCTION

In a previous paper (1) we presented a theory describing the capillary rise between two closely spaced rigid cylinders of circular cross section. The model of parallel cylinders was chosen to represent certain fibrous systems such as woven textiles. The structural element in these fabrics, which is of the greatest importance in terms of capillarity, is the yarn, consisting of many individual fibers. For this reason the theory of capillary rise has been extended to assemblies of three and four equidistant parallel cylinders. The multicylinder system will then be treated as an aggregate of a large number of either of these basic units.

As before (1), only equilibrium conditions will be considered. It is also assumed again that the contact angle \( \theta \) is the same along the whole solid/liquid/air boundary.

THEORY

1. Capillary Rise between Three Equidistant Parallel Cylinders of Equal Radius

When three wettable identical rods in contact are placed in a liquid, the liquid will rise in the triangular channel between the cylinders and attain a certain equilibrium configuration. Figure 1a shows the horizontal cross section through the rods; Fig. 1b gives the vertical section through DB. The vertical dashed line corresponds to point E in Fig. 1a, and the solid line to F. The curved lines indicate the approximate shape of the liquid surface in this section. Between each pair of adjacent rods the liquid rises to infinity as described before (1), and in the center 0 of the channel a meniscus is formed at a height \( z_3 \) above the horizontal bulk surface. When the rods are separated, so that the distance between each pair is \( 2d \) (Fig. 1c), the vertical section through EF shows a high meniscus at E at a height \( z_2 \) above the horizontal bulk surface. When the rods are separated, so that the distance between each pair is \( 2d \) (Fig. 1c), the vertical section through EF shows a high meniscus at E at a height \( z_2 \) above the bulk surface (Fig. 1d), and a lower meniscus in the center of the channel at a height \( z_3 \) which is lower than in Fig. 1b. If we anticipate our results, further separation leads to a decrease in both \( z_2 \) and \( z_3 \), with \( z_3 \) falling much faster than \( z_2 \). This leads to a situation, at a certain value of \( d \), where the two menisci "merge" and can no longer be distinguished, in which case the vertical section looks qualitatively
Fig. 1. (a) Horizontal cross section through three touching cylinders. (b) Schematic profile of liquid interface in vertical section through DB in (a). (c) Horizontal cross section through three separated cylinders. (d) Interfacial profile in vertical section through EF in (c) for $d < d_{tr}$. (e) Same for $d > d_{tr}$.

as in Fig. 1e. Then the capillary rise can be described by one meniscus height, $z_2$. The goal of this study is to predict $z_2$ and $z_3$ as a function of the distance $2d$ and the contact angle $\theta$, measured through the liquid phase. This can be conveniently done only if it is assumed that the meniscus height is always much greater than the cylinder radius $r$, i.e.,

$$z_3 \gg r.$$

Then the dimensions of the menisci are negligible compared to the heights $z_2$ and $z_3$, and their weight will be negligible compared to that of the liquid columns below the menisci. Therefore, limiting solutions will be obtained, which become exact when $z_3/r$ approaches infinity.

In the detailed analysis we shall distinguish two regions of $d$: $d < d_{tr}$ and $d > d_{tr}$, where $2d_{tr}$ is the distance between the cylinders at which the transition occurs between the type of meniscus of Fig. 1d and that of Fig. 1e. It can be anticipated that $d_{tr}$ depends on the contact angle $\theta$.

a. $d < d_{tr}$. At these relatively small separations, the high saddle-shaped meniscus at $z_2$ is identical to that between two cylinders. In other words, the third cylinder does not affect the height of this meniscus. Therefore, $z_2$ can be obtained as a function of $d/r$ and $\theta$ from reference 1, where $z_2$ was expressed in the form

$$z_2 = \frac{r}{R_2},$$

where $R_2$ is the radius of curvature of the essentially vertical liquid surface just below the meniscus, and

$$c = \frac{\rho g}{\gamma},$$

where $\rho$ is the density of the liquid, $\gamma$ is its surface tension, and $g$ is the acceleration due to gravity. Reference 1 (a) gives $r/R_2$ as a function of $d/r$ for various values of $\theta$.

The lower meniscus requires further investigation. One might propose, rather arbitrarily, that a reasonable estimate of $z_3$
should be obtained by considering the rise in a cylindrical capillary of radius \( r_i \), where \( r_i \) is the radius of the inscribed circle in the triangular channel. As \( r_i \) is given by

\[
\frac{r_i}{r} = \frac{d}{\sqrt{3} r} + \frac{2}{\sqrt{3}} - 1 = 1.1547 \frac{d}{r} + 0.1547, \tag{3}
\]

the corresponding capillary rise \( z_i \) would be given by

\[
z_{i\rho g} = 2 \gamma \cos \theta / r_i \tag{4}
\]

or

\[
\frac{z_i}{r} = \frac{1}{cr^2} \cdot \frac{2 \cos \theta}{1.1547(d/r) + 0.1547}. \tag{5}
\]

It will be shown, however, that this estimate of \( z_i \) deviates appreciably from the actual meniscus height.

A much better estimate of \( z_i \) is obtained by a reasoning very similar to that used in reference 1 for the two-cylinder problem. Between \( z_2 \) and \( z_3 \) in Fig. 1d the liquid surfaces of the "finger," pointing upward, are symmetrical with respect to the line \( E \). They are also essentially vertical, so that their total curvature is characterized by the radius \( R \) of the circular arcs in the horizontal cross section as shown in Fig. 2, which is drawn for a finite contact angle \( \theta \). The angle between the line connecting the centers of two adjacent cylinders and the radius pointing at the solid/liquid/air boundary is \( \alpha \). As in the two-cylinder problem (1), \( R \) is related to \( \alpha \) through

\[
R = \frac{1 + d/r - \cos \alpha}{\cos (\theta + \alpha)}. \tag{6}
\]

\( R \), and therefore \( \alpha \), are determined by the height \( z \) above the bulk surface (1) according to

\[
\frac{z}{r} = \frac{1}{cr^2} R. \tag{7}
\]

Equations [6] and [7] hold for the outer surfaces for any value of \( z < z_2 \), except for the region immediately above the bulk surface. For the inner surfaces, however, they apply only for \( z_2 > z > z_3 \). At \( z_3 \) the third cylinder starts to exert its influence on the shape of the liquid surface, which bends sharply to form the meniscus in the center of the channel. At this level, \( \alpha = \alpha_3 \) and \( R = R_3 \).

The problem can now be solved by equating the weight of the liquid column inside and directly below the perimeter \( STUVWX \), and the surface tension force acting along this perimeter. The area of the column is

\[
A = 3\pi^2[(R_3/r)^2 \{ \cos^2 (\theta + \alpha_3)/\sqrt{3}
+ \pi/2 - (\theta + \alpha_3) - \sin (\theta + \alpha_3)
\cdot \cos (\theta + \alpha_3) \} + (R_3/r)^2 \cdot (2 \cos \alpha_3 \cos (\theta + \alpha_3) )]
+ \cos^2 \alpha_3 / \sqrt{3} + \alpha_3 - \pi/6 - \cos \alpha_3 \cos \alpha_3,
\]

and the weight of the liquid column is, therefore,

\[
F_1 = z_{i\rho g} A. \tag{9}
\]

The surface tension force is given by

\[
F_2 = (TU + VW + XS)\gamma + (ST + UV + WX)\gamma \cos \theta
= 3\pi r [2(\pi/2 - \alpha_3 - \theta)(R_3/r)
+ (\pi/3 - 2\alpha_3) \cos \theta]. \tag{10}
\]

Substituting for \( z_3 \) and \( A \) in Eq. [9] according to [7] and [8], equating \( F_1 \) and \( F_2 \), and

rearranging, one obtains

\[ \frac{(R_3/r)^2}{(R_3/r)^2} \cos^2 (\theta + \alpha_3) / \sqrt{3} - \pi/2 \]
\[ + (\theta + \alpha_3) - \sin (\theta + \alpha_3) \]
\[ \cdot \cos (\theta + \alpha_3) \]
\[ + (R_3/r) \cdot [2 \cos \alpha_3 \cos (\theta + \alpha_3) / \sqrt{3} \]
\[ - 2 \sin \alpha_3 \cos (\theta + \alpha_3) \]
\[ + 2(\alpha_3 - \pi/6) \cos \theta \]
\[ + \cos^2 \alpha_3 / \sqrt{3} + \alpha_3 - \pi/6 \]
\[ - \sin \alpha_3 \cos \alpha_3 = 0. \]

[11]

From this quadratic equation \( R_3/r \) can be evaluated as a function of \( \alpha_3 \), and the corresponding value of \( d/r \) can be obtained from Eq. [6]. As in reference 1, however, we chose \( d/r \) as the independent variable. The corresponding values of \( \alpha_3 \) and \( R_3/r \) were calculated with an electronic computer by a method of successive approximation. The capillary rise \( z_1 \) is inversely proportional to \( R_3/r \) according to Eq. [7].

When this procedure is followed, one finds that at a certain value of \( d = d_0 \), the curves for \( r/R_2 \) and \( r/R_3 \) (or \( z_2/r \) and \( z_3/r \)) intersect. At larger separations a different treatment becomes necessary.

b. \( d > d_0 \). In this region the meniscus will have a shape as shown schematically in Fig. 1c. As long as \( z_2 >> r \) the dimensions of the meniscus region are still negligible, and the surfaces outside and below the meniscus are essentially vertical and cylindrical. The analysis becomes then almost identical to that given in reference 1 for two cylinders.

A horizontal cross section just below the meniscus looks like Fig. 3. One can now state that the weight of the liquid column directly below \( STUVWX \) (shaded in Fig. 3) must be supported by the surface tension forces along the arcs \( ST, UV, VW, WX \), and \( XS \). The weight \( F_1 \) of the liquid column is again given by Eq. [9], where \( A \) equals

\[ A = 3\pi^2 (R_3/r)^2 \cos^2 (\theta + \alpha_3)/ \sqrt{3} \]
\[ - \pi/2 + (\theta + \alpha_3) \]
\[ + \sin (\theta + \alpha_3) \cos (\theta + \alpha_3) \]
\[ + (R_3/r) \cdot [2 \cos \alpha_3 \cos (\theta + \alpha_3) / \sqrt{3} \]
\[ \cdot \cos^2 \alpha_3 / \sqrt{3} - \alpha_3 - \pi/6 \]
\[ + \sin \alpha_3 \cos \alpha_3. \]

The arcs \( TU, VW \), and \( XS \) contribute an upward force

\[ F_{21} = 6\gamma r(\alpha_3 + \pi/6) \cos \theta, \]

[13]

whereas the free liquid surfaces along arcs \( ST, UV, \) and \( WX \) give rise to a downward force

\[ F_{22} = -6\gamma r(\pi/2 - \alpha_3 - \theta)R_3/r. \]

[14]

Substituting for \( z_3 \) and \( A \) in Eq. [9] according to [7] and [12], equating \( F_1 \) and \( (F_{21} + F_{22}) \), and rearranging, one finally obtains the equilibrium condition

\[ (R_3/r)^2 \cos^2 (\theta + \alpha_3) / \sqrt{3} + \pi/2 \]
\[ - (\theta + \alpha_3) + \sin (\theta + \alpha_3) \]
\[ \cdot \cos (\theta + \alpha_3) \]
\[ + (R_3/r) \cdot [2 \cos \alpha_3 \cos (\theta + \alpha_3) / \sqrt{3} \]
\[ + 2 \sin \alpha_3 \cos (\theta + \alpha_3) \]
\[ - 2(\alpha_3 + \pi/6) \cos \theta + \cos^2 \alpha_3 / \sqrt{3} \]
\[ - \alpha_3 - \pi/6 + \sin \alpha_3 \cos \alpha_3 = 0. \]

[15]

The first paragraph following Eq. [11] is applicable here also. Results will follow the theoretical section.

2. CAPILLARY RISE BETWEEN FOUR VERTICAL EQUIDISTANT CYLINDERS OF EQUAL RADIUS

A horizontal cross section through the cylinders is shown in Fig. 4a. The distance between two adjacent cylinders is \( 2d \). This system has two different types of symmetry planes, i.e., \( GH \) and \( BD \). At small values of \( d/r \) the vertical section through \( GH \) shows two high saddle-shaped menisci (at \( E \) and \( F \)) and a low meniscus in the center \( O \) of the channel (Fig. 4b). The liquid surface in the vertical section through \( BD \) is obviously of a simpler type (Fig. 4c). The curvature of the surface at \( O \) must be the same in both sections.
As discussed above for three cylinders, with increasing \( d/r \) the height \( z_2 \) decreases more rapidly than \( z_4 \) until at \( d = d_{tr} \) the menisci merge and one meniscus is formed of the type shown in Fig. 4d. The section through \( BD \) will remain qualitatively similar to Fig. 4c.

The analysis is almost identical to that given above for three cylinders. Equations [6] and [7] remain valid, and the range of \( d/r \) is again divided into two subranges: \( d < d_{tr} \) and \( d > d_{tr} \). In the first case \( z_2/r \) is again given in reference 1; and for the lower meniscus the equivalent of Eq. [11] is

\[
(R_i/r)^2\sqrt{\cos^2(\theta + \alpha_i) - \pi/2 + (\alpha_i + \theta)} - \sin(\theta + \alpha_i) \cos(\theta + \alpha_i) + (R_i/r)^2\cos \alpha_4 \cos(\theta + \alpha_4) - 2 \sin \alpha_4 \cos(\theta + \alpha_4) + 2(\alpha_4 - \pi/4) \cos \theta + \cos^2 \alpha_4 - \pi/4 + \alpha_i - \pi/4 - \sin \alpha_i \cos \alpha_i = 0.
\]

For \( d > d_{tr} \) the equivalent of Eq. [15] becomes

\[
(R_i/r)^2\sqrt{\cos^2(\theta + \alpha_i) + \pi/2 - (\theta + \alpha_i) + \sin(\theta + \alpha_i) + (R_i/r)^2\cos \alpha_4 \cos(\theta + \alpha_4) + 2 \sin \alpha_4 \cos(\theta + \alpha_4) - 2(\alpha_4 + \pi/4) \cos \theta - \cos^2 \alpha_4 - \pi/4 + \alpha_i - \pi/4 - \sin \alpha_i \cos \alpha_i = 0.
\]

Finally, the equivalent equations for [3] and [5], referring to the capillary rise in a cylindrical tube of radius \( r_i \) (radius of the inscribed circle), are

\[
\frac{r_i}{r} = \sqrt{\frac{d}{r}} + \sqrt{2} - 1
\]

\[
= 1.4142 \frac{d}{r} + 0.4142
\]

and

\[
\frac{z_i}{r} = \frac{1}{\sqrt{2}} \cdot \frac{2 \cos \theta}{1.4142(d/r) + 0.4142}.
\]

3. Capillary Rise in a Multicylinder System

In Fig. 5 two possible regular arrangements are given for a multicylinder system. In Fig. 5a there is a hexagonal arrangement, built up of the triangular units discussed above; in Fig. 5b a square arrangement is chosen, consisting of a succession of square "cells."

With the knowledge of the capillary behavior in the unit cells the meniscus height in both multicylinder systems can be readily obtained. One must again distinguish two regions of cylinder/cylinder separation: $d < d_{tr}$ and $d > d_{tr}$, where $d_{tr}$ has the same value as for the unit cell.

a. $d < d_{tr}$. In this region the results obtained for the unit cells are directly applicable to the multicylinder systems (a) and (b). In other words, there are high saddle-shaped menisci at a height $z_2$ between each pair of adjacent cylinders and low menisci in the centers of the channels at a height $z_3$. These heights are identical to those calculated for the unit cells, because the cylinders outside a cell may be assumed to have no effect on the shape of the interfaces inside that cell. The different cells are "insulated" from each other by the high menisci.

Figure 6 shows the liquid profile in the vertical section through $AB$ in Fig. 5 for $d < d_{tr}$. Points $E$ are located midway between adjacent cylinders.

b. $d > d_{tr}$. At these relatively large separations the results obtained for the unit cells are not directly applicable. The reason for this is the change in the symmetry of the system. The vertical lines through $E$ in Fig. 5a and 5b are lines of symmetry in the vertical section through $AB$, whereas they were not in the unit cells for $d > d_{tr}$ (cf. Figs. 1e and 4d). In the vertical section $AB$ the liquid surface describes a periodic wave, the amplitude of which is negligible compared to $z_3$ (Fig. 7). Because of this additional element of symmetry the problem becomes very simple.

If one considers a single unit cell of Fig. 5a, the liquid inside the cell must be supported exactly by the surface tension along the arcs $ST$, $UV$, and $WX$ in Fig. 8, as the
surface tension is directed horizontally between $T$ and $U$, $V$ and $W$, and $X$ and $S$.

The weight of the liquid column is again given by Eq. [9], where

$$A = r^2[(1 + d/r)^2 \sqrt{3} - \pi/2]. \quad [20]$$

The upward surface tension force is simply

$$F_2 = \pi r \gamma \cos \theta. \quad [21]$$

Equating $F_1$ and $F_2$, and rearranging, one finds

$$\frac{z_3}{r} = \frac{1}{cr^2} \left(1 + \frac{d}{r}\right)^2 \sqrt{3} - \pi/2. \quad [22]$$

Similarly, for $d > d_{tr}$ in the square multicylinder arrangement

$$A = r^2[4(1 + d/r)^2 - \pi], \quad [20a]$$

$$F_2 = 2\pi r \gamma \cos \theta, \quad [21a]$$

so that

$$\frac{z_4}{r} = \frac{1}{cr^2} \sqrt{\frac{\pi \cos \theta}{2(1 + d/r)^2 - \pi/2}}. \quad [22a]$$

Formally, the last terms in Eqs. [22] and [22a] stand for $r/R$ in Eq. [7], although in this case $R$ has no physical meaning other than the reciprocal mean curvature of the meniscus.

The preceding treatment for $d > d_{tr}$ applies, strictly speaking, only to assemblies of an infinite number of cylinders. When the number is finite, one can expect deviations in the outer layer(s) of the assembly, where the required symmetry conditions are destroyed.

RESULTS AND DISCUSSION

THREE CYLINDERS

When $d < d_{tr}$ the system can be described by the heights of two menisci, i.e., $z_2$ and $z_3$. For $d > d_{tr}$ only one meniscus remains, which is characterized by its height $z_3$. Results on $z_2$ have been presented in detail in reference 1 by tabulating $r/R_2$ as a function of $d/r$ for various contact angles $\theta$. To maintain uniformity in data presentation, results for $z_3$ are given in Table I in the same form, i.e., as $r/R_3 (= cr^2 z_3/r)$. Also included are the corresponding values of $\alpha_3$, which give an immediate impression of the shape of the cylindrical interfaces adjoining the meniscus. The dashed lines through the columns of Table I indicate the approximate value of $d_{tr}/r$. More accurate values are given in Table II. The relative meniscus heights are plotted in Fig. 9 for $\theta = 0^\circ$ only. It is reassuring that the two branches at $d < d_{tr}$ and the single branch at $d > d_{tr}$ pass through one point. The dashed line represents the estimated capillary rise, based on the radius of the inscribed circle (see Eq. [5]). It is clear that this approach is not reliable and overestimates the true meniscus height $z_3$.

FOUR CYLINDERS

The results for this case are presented in a similar way. Table III gives $r/R_4$ and accurate values of $d_{tr}/r$ are entered in Table II. Figure 10 shows the relative meniscus heights for $\theta = 0^\circ$. The estimated capillary rise, derived from the radius of the inscribed circle according to Eq. [19], is also included.

MULTICYLINDER SYSTEMS

No additional numerical data are necessary to describe this case. For $d < d_{tr}$,
TABLE I
PARAMETERS CHARACTERIZING THE CAPILLARY RISE BETWEEN THREE CYLINDERS

<table>
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<th>( \theta = 15^\circ )</th>
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<td>9.55</td>
</tr>
<tr>
<td>0.70</td>
<td>0.3056</td>
<td>9.99</td>
<td>0.2884</td>
<td>9.04</td>
<td>0.5834</td>
<td>8.28</td>
</tr>
<tr>
<td>0.80</td>
<td>0.1925</td>
<td>8.83</td>
<td>0.1516</td>
<td>8.30</td>
<td>0.4013</td>
<td>7.60</td>
</tr>
<tr>
<td>0.90</td>
<td>0.1001</td>
<td>7.93</td>
<td>0.0732</td>
<td>7.86</td>
<td>0.2235</td>
<td>7.15</td>
</tr>
<tr>
<td>1.00</td>
<td>0.0461</td>
<td>6.95</td>
<td>0.0143</td>
<td>7.36</td>
<td>0.1054</td>
<td>6.55</td>
</tr>
</tbody>
</table>

Tables I and III can be used, depending on whether a hexagonal or square arrangement is considered. For \( d > d_r \), the simple equations [22] and [22a] can be used directly.

The tables permit the evaluation of \( r/R_3 \) and \( r/R_4 \) when \( d/r \) and \( \theta \) are known. The corresponding height of the meniscus is then obtained with Eq. [7] if the capillary constant \( c \) and the radius of the cylinders are known also.

Tables I and III contain only a fraction of the data generated. The tables have been reduced in size for the purpose of space saving. This means that in certain ranges of \( d/r \) the intervals are somewhat too large for accurate interpolation of the important parameter \( r/R \) from which the capillary rise is obtained. As noted in reference 1, the parameter \( d/R (= d/r \cdot r/R) \) can be helpful in this respect, as it does not vary as strongly as \( r/R \). Therefore, a column with values of \( d/R \) for each contact angle can be added by the interested reader. Then an interpolated value of \( d/R \) is obtained for given \( d/r \), and \( r/R \) is calculated from \( r/R = (d/R)/(d/r) \).

CAPILLARY PHENOMENA IN ASSEMBLIES OF PARALLEL CYLINDERS. II

As in the two-cylinder system (1), the tables terminate at a certain value of \( d/r = (d/r)_{\text{max}} \), where the theory predicts zero capillary rise. At this distance of separation the arcs ST, UV, and WX in Fig. 3 become straight lines \( (R_3 = \infty) \). This occurs when \( \alpha_3 = \pi/2 - \theta \). By a reasoning similar to that given for two cylinders (1) one finds for three cylinders

\[
(d/r)_{\text{max}} = (2\pi/3 - \theta) \cos \theta + \sin \theta - 1 \tag{23}
\]

and for four cylinders

\[
(d/r)_{\text{max}} = (3\pi/4 - \theta) \cos \theta + \sin \theta - 1. \tag{24}
\]

It should be realized that for \( d/r > (d/r)_{\text{max}} \) some capillary rise will take place, but the meniscus height will then be of the same order of magnitude as—or smaller than—the radius of the cylinders. In other words, in this range of \( d/r \) the above theory is useless, whereas for \( d/r < (d/r)_{\text{max}} \) the theory is applicable provided the cylinders are sufficiently thin to guarantee that \( z_3 \) or \( z_4 \gg r \).

The multicylinder system chosen in this study seems to be the most reasonable model for a textile yarn. Nevertheless, in such a real system further complications arise. In the first place, the arrangement will not be as regular as shown in Fig. 5, so that there will be a distribution of high and low menisci in the yarn. The hexagonal arrangement will be a more favorable packing, however. A further complication is the deviation from parallelism that will undoubtedly occur locally. Also, the assumption of circular cross section, although reasonable for many synthetic materials, certainly does not apply to fibers such as cotton. Finally, capillary rise itself will give rise to changes in the yarn because of capillary forces which tend to pull the fibers together, starting from the outer layer.

CAPILLARY RISE IN NONCIRCULAR CLOSED CAPILLARIES

The reasoning leading to the capillary rise in the channel between three or four cylinders can also be used to calculate the capillary rise in closed capillaries of noncircular cross section (e.g., an equilateral triangle or square).

TRIANGULAR CROSS SECTION (Fig. 11a)

The length of each side of the triangle \( \triangle ABC \) is \( 2a \). Although the problem can be solved readily for any contact angle, only zero \( \theta \) will be considered here. In general terms one can say that the meniscus will have its lowest point in the center \( O \) of the channel, while the liquid rises in the corners to infinite height. This is shown in Fig. 11b.

TABLE II

<table>
<thead>
<tr>
<th>( \theta )</th>
<th>Three cylinders</th>
<th>Four cylinders</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \theta )</td>
<td>( d_{1/2}/r )</td>
<td>( r/R )</td>
</tr>
<tr>
<td>0°</td>
<td>0.079</td>
<td>7.05</td>
</tr>
<tr>
<td>15°</td>
<td>0.075</td>
<td>7.04</td>
</tr>
<tr>
<td>30°</td>
<td>0.065</td>
<td>6.92</td>
</tr>
<tr>
<td>45°</td>
<td>0.052</td>
<td>6.45</td>
</tr>
<tr>
<td>60°</td>
<td>0.035</td>
<td>5.49</td>
</tr>
<tr>
<td>75°</td>
<td>0.016</td>
<td>3.72</td>
</tr>
</tbody>
</table>

which represents a vertical section through AD. From just above the lower meniscus upward, the surface of the liquid in the corner is very close to vertical and cylindrical, i.e., its curvature is characterized by the radius $R$ in the horizontal cross section (Fig. 11a), where $R$ decreases linearly with $R = R_s$, and the surface bends sharply to the height $z$ above the bulk surface according to

$$z = 1/cR$$

[25]

This relation holds down to a level $z_s$, where $R = R_s$, and the surface bends sharply to form the meniscus the dimensions of which are negligible compared to $z_s$ provided the capillary is sufficiently narrow. At $z_s$ the
The surface tension force is
\[ F_2 = \gamma [2 \pi R_5 + 6(a - b)] \]
\[ = 2a \gamma \left[ \frac{\pi R_5}{a} + 3 \left(1 - \frac{R_3}{a} \sqrt{3}\right)\right]. \]  
Equating [27] and [29], substituting for \( z \) according to [26], and solving for \( R_3/a \) leads to
\[ R_3/a = 0.3248, \]  
which yields
\[ b/a = 0.3248 \tan 60^\circ = 0.5626. \]  
Substituting for \( R_3/a \) in Eq. [26] gives the final result for the capillary rise:
\[ z_3/a = 3.079/\gamma a^2 \]  
or
\[ z_3 = 3.079 \gamma /\rho g a. \]  
**Square Cross Section**

In a completely analogous manner one finds for the capillary rise \( z_4 \) in a capillary of square cross section, each side having a length \( 2a \):
\[ R_4/a = b/a = 0.5301 \]  
and
\[ z_4/a = 1.886/\gamma a^2 \]  
or
\[ z_4 = 1.886 \gamma /\rho g a. \]  
If one would assume the capillary rise in these capillaries to be the same as that in a circular capillary of radius \( r_i \), i.e., the radius of the inscribed circles, one would find for the triangular capillary
\[ z_3/a = 3.464/\gamma a^2 \]  
and for the square capillary
\[ z_4/a = 2/\gamma a. \]  
These equations, therefore, overestimate the capillary rise by about 11% and 6%, respectively.

**Estimation of Errors**

All solutions presented in this as well as the previous paper (1) are limiting solutions, Journal of Colloid and Interface Science, Vol. 30, No. 3, July 1969
and it has been repeatedly emphasized that the condition for their applicability is that \(z/r \gg 1\). In this respect, the degree of sophistication of these equations is comparable to that of the simple equation for the capillary rise \(z\) in a capillary of circular cross section of radius \(r\), i.e.,

\[z \rho g = 2\gamma/r\]

or

\[z/r = 2/cr^2, \quad [39]\]

which is also valid only when \(z/r \gg 1\). Whereas these approximations hold very well in narrow channels, corrections become necessary when the tube dimensions are comparable to \(\sigma^{-1/2}\). These corrections have been obtained for the circular capillary (2-4), where the meniscus is always axially symmetric. For the systems described in this study refinements are much more difficult to obtain because of the lower degree of symmetry. However, in a number of cases a semiquantitative error analysis can be made.

We refer to \(z_3\) and \(z_4\) between three or four cylinders when \(d < d_v\), and to \(z_3\) and \(z_4\) in triangular or square closed capillaries. In these four cases the meniscus is a convex surface, i.e., its two curvatures have everywhere the same sign. Although these menisci are far from spherical, it seems reasonable, as a first approximation, to assume that Sugden's correction factors (4) can be applied to these systems.

Thus, for zero contact angle, let us define the equivalent radius \(r_e\) of the channel as the radius of a closed circular capillary with the same limiting capillary rise. For example, for the lower meniscus between three cylinders, Eqs. [7] and [39] then lead to

\[z = \frac{1}{cr} \cdot \frac{r}{R_3} = \frac{2}{cr_e}\]

or

\[r_e/r = \frac{2}{r/R_3}. \quad [40]\]

The corrected capillary rise in the equivalent capillary is given by

\[z = \frac{2}{cr_a} \cdot Q, \quad [41]\]

where \(Q\) is Sugden's correction factor, given by

\[Q = r_e/a,\]

where \(1/b\) is the mean curvature in the lowest point of the meniscus. Sugden has tabulated \(Q\) as a function of \(r_e/a\), where \(a\) is Laplace's capillary constant

\[a = (2/c)^{1/2}.\]

Assuming that the same correction factor applies, at least approximately, to \(z_3\) and \(z_4\) between cylinders, one obtains

\[z_3/r = \frac{1}{cr^2} \cdot R_3 \cdot Q. \quad [42]\]

Thus, for certain \(d/r < d_v/r\) and \(\theta = 0^\circ\), Table I gives \(r/R_3\), which yields \(r_e\) from Eq. [40]. If \(c\) and, therefore, \(a\) are known, \(Q\) is obtained from reference 4.

For example, for \(r = 0.1\ cm, d = 0.005\ cm,\) and \(c = 14\ cm^{-2}\) (water), one finds \(d/r = 0.05, r/R_3 = 8.196, r_e = 0.0244\ cm,\) and \(r_e/a = 0.0646.\) From Sugden's table one finds for this case \(Q = 0.9985,\) so that the limiting theory is wrong by less than 0.2%.

If all dimensions are ten times greater (\(r = 1\ cm, d = 0.05\ cm\)), however, one finds \(r_e/a = 0.646\) and \(Q = 0.8787,\) indicating that the correction factor amounts to about 12%.

In the former case \(z_3/r = 58.46;\) in the latter \(z_3/r = 0.514.\)

Similar considerations apply to \(z_4\) between four cylinders when \(d/r < d_v/r\) and to \(z_3\) and \(z_4\) in triangular and square closed capillaries.

For \(z_2\) between two cylinders or between a cylinder and a plate (1), and for \(z_3\) and \(z_4\) in systems with \(d/r > d_v/r,\) we have not yet been able to develop a similar error analysis because of the complexity of the meniscus shape. There is experimental evidence, however, that in the case of \(z_3\) the various correction factors that have to be considered, such as the weight and dimensions of the meniscus and the deviation of the adjoining surfaces from the vertical, tend to cancel to an appreciable extent, so that the condition \(z_3/r \gg 1\) may be overly severe. For example, in a cylinder/plate combination with \(r = 0.634\ cm, d = 0.02016\ cm, c = 27.5\ cm^{-2},\) and \(\theta = 0^\circ,\) we measured a value of \(z_2\) of 1.33 cm, which is only about 3% lower than

the value predicted by the limiting theory of reference 1, even though \( z_2/\gamma \) was only 2.10. Further experiments are in progress and will hopefully be reported in a future communication.

ACKNOWLEDGMENT

I thank Dr. E. D. Goddard for stimulating discussions and the Lever Brothers Company for permission to publish this paper.

REFERENCES

1. PRINCE, H. M., *J. Colloid and Interface Sci.*, 30, 69, (1969); (a) *ibid.*, Table I.