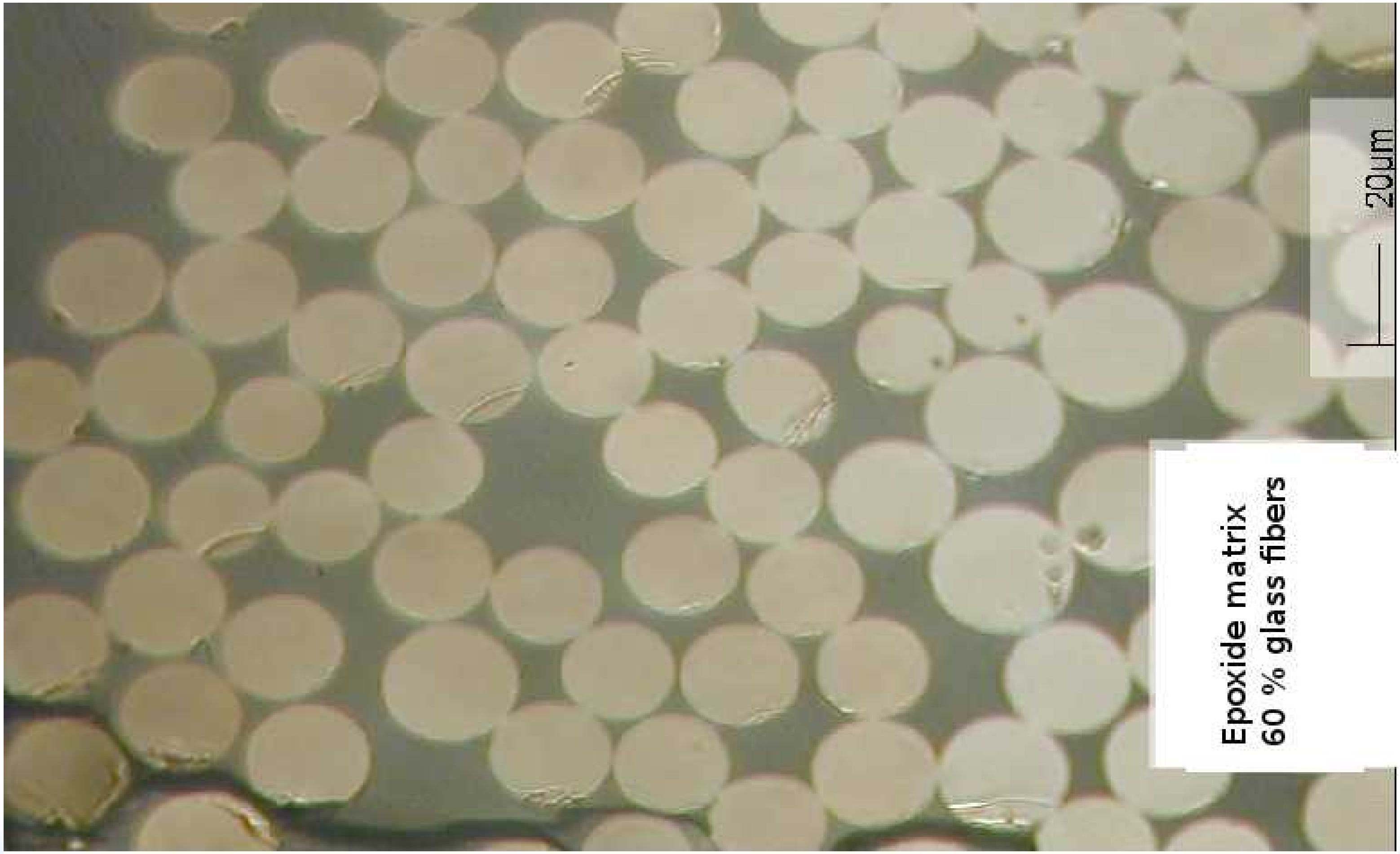
Halpin – Tsai model



The essence of the model

Very versatile model allows calculation of the various elastic constants of the composite under tensile loading in different directions.

For the nanocomposites is very often used.

Results for Young's modulus are between Voigt's and Reuss model.

Used symbols

- Because the Halpin-Tsai model is quite general, uses a slightly different symbols than we have used:
- E_T ... Young's modulus of the composite
- E_f, E_m ... Young's modulus of fibers (dispersion) or matrix
- a or b ... dimension of the cross section of dispersion paralel or perpendicular to the direction of applied tensile force

Relation for Young's modulus

There is halfempirical aproximation, relation for Young's modulus E_T :

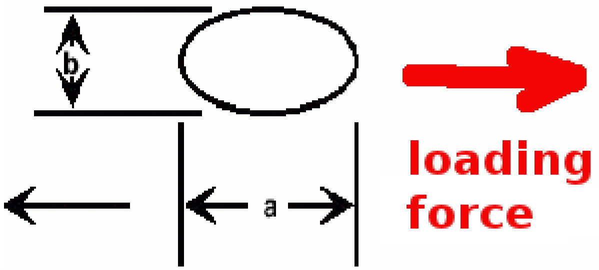
$$\frac{E_T}{E_m} = \frac{1+\xi\eta V_f}{1-\eta V_f}$$

For constants in relation applies

$$\eta = \frac{\frac{E_f}{E_m} - 1}{\frac{E_f}{E_m} + \xi}$$

$$\xi = 2\left(a/b\right)$$

Constant ξ counts the lateral dimensions of the dispersion as shown:

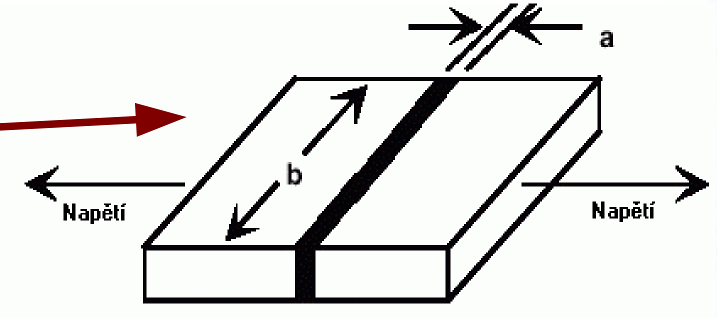


Halpin-Tsai relation by transversal plates

The arrangement according to the that picture,

a → 0 and b → ∞

and therefore ξ = 0



Resulting relation :

$$\frac{E_T}{E_m} = \frac{E_f}{V_f E_m + V_m E_f}$$

Acording to Reuss model

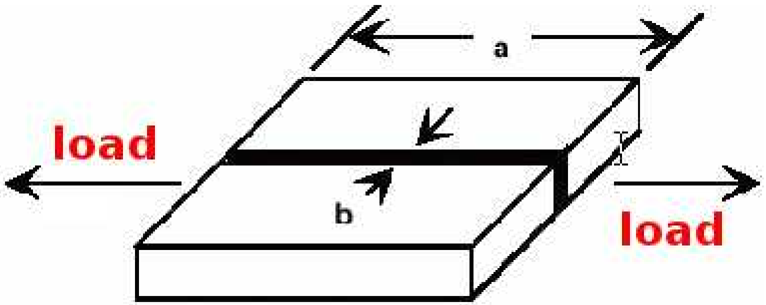
- Halpin – Tsai realtion :

$$\frac{E_T}{E_m} = \frac{1}{1 - \eta V_f}$$

$$\eta = \frac{E_f - E_m}{E_f}$$

Halpin – Tsai relation for paralel plates, or fibers

The arrangement according to the that picture,
$a \to \infty$, $b \to 0$
a tedy $\xi \to \infty$



By constant η is in denominator ξ, that goes to infinity, constant η must go to zero.
Product $\xi * \eta$ we can write

$$\xi * \eta = \xi * (E_f/E_m - 1)/(E_f/E_m + \xi)$$

$$= (E_f/E_m - 1)/((E_f/E_m)/\xi + 1)$$

Because $(E_f/E_m)/\xi$ goes to zero, must be

$$\xi * \eta = E_f/E_m - 1$$

Resulting relation

- Resulting relation for Young's modulus is
  $E_T / E_m = (1 + (E_f/E_m - 1) * v_f)/(1+0)$
  and after multiplying with E_m
  $E_T = E_m + E_f * v_f - E_m * v_f$,
  because $1 - v_f = v_m$, it is
  $$E_T = E_m * v_m + E_f * v_f$$
- Relulting relation is the same as ný Voigt's model.

Comparison of models

- As is evident from the above, it is possible to consider Reuss and Voigt's model as a special case of model Halpin - Tsai.
- Model Halpin - Tsai is a very general model for composite with arbitrary shape dispersion, with load in the main direction or perpendicular to the main direction.
- In Halpin - Tsai model, there are also relationships for the other elastic constants

General relations

P_c, P_m, P_f … any elastic constant of composite, matrix, dispersion

f … volume concentration of dispersion

$$P_c = P_m \left(\frac{1 + \zeta \eta f}{1 - \eta f} \right) \qquad \eta = \frac{\left(\frac{P_f}{P_m} \right) - 1}{\left(\frac{P_f}{P_m} \right) + \zeta}$$

Coefficient size ζ

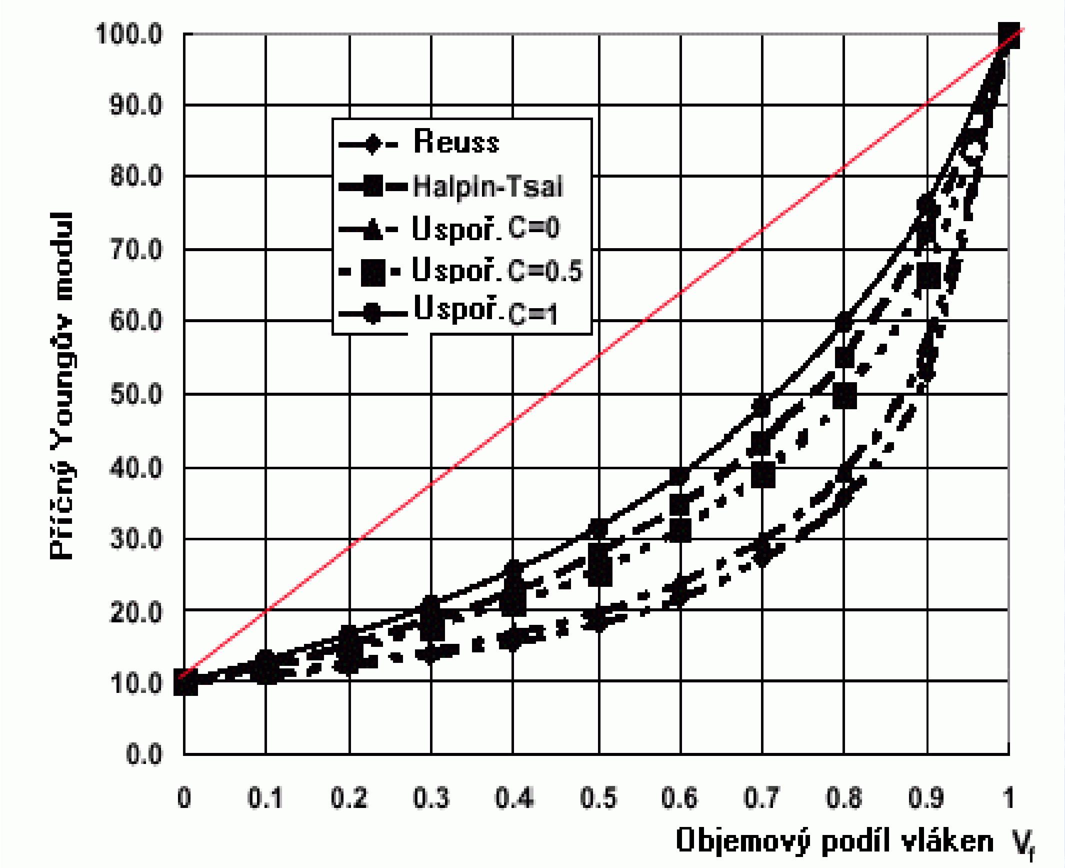
Geometry	Ex	Ey	ν	G
Aligned continuous fibres	$fE_f + (1-f)E_m$	$\frac{E_f E_m}{fE_m + (1-f)E_f}$ or $\zeta = 2 + 40f^{10}$	$f\nu_f + (1-f)\nu_m$	$\zeta = 1 + 40f^{10}$ or $G_m\left(\frac{G_f(1+f) + G_m(1-f)}{G_m(1+f) + G_f(1-f)}\right)$
Spherical particles	$\zeta = 2 + 40f^{10}$	$\zeta = 2 + 40f^{10}$	$f\nu_f + (1-f)\nu_m$	$\zeta = 1 + 40f^{10}$
Oriented short fibres	$l < l_c \quad E_m\left(1-f\left(1-\frac{l}{2d}\right)\right)$ $l \geq l_c \quad fE_f\left(1-\frac{l_c}{2l}\right) + (1-f)E_m$	$\zeta = 2 + 40f^{10}$	$f\nu_f + (1-f)\nu_m$	$\zeta = 1 + 40f^{10}$
Oriented plates	$\zeta = 2\left(\frac{l}{t}\right) + 40f^{10}$	$\zeta = 2\left(\frac{w}{t}\right) + 40f^{10}$	$f\nu_f + (1-f)\nu_m$	$\zeta = \left(\frac{l+w}{2t}\right)^{1.73} + 40f^{10}$
Oriented whiskers	$\zeta = 2\left(\frac{l}{d}\right) + 40f^{10}$	$\zeta = 2 + 40f^{10}$	$f\nu_f + (1-f)\nu_m$	$\zeta = \left(\frac{l}{d}\right)^{1.73} + 40f^{10}$

Comparison of perpendicular models

$E_m = 10$, $E_f = 100$

Red .. Voigt's model

Reuss model is lowest,
model with coefficient of arrangement
C = 1 is higest.
Halpin Tsai is in the middle
– very real,
for short fibers and particles too



Comparison model with experiment

- **Glass Fibers:**
  $E_f = 73$ GPa, $\nu_f = 0.22$, $c_f = 0.55$
- **Epoxy Matrix:**
  $E_f = 3.5$ GPa, $\nu_f = 0.35$, $c_f = 0.45$

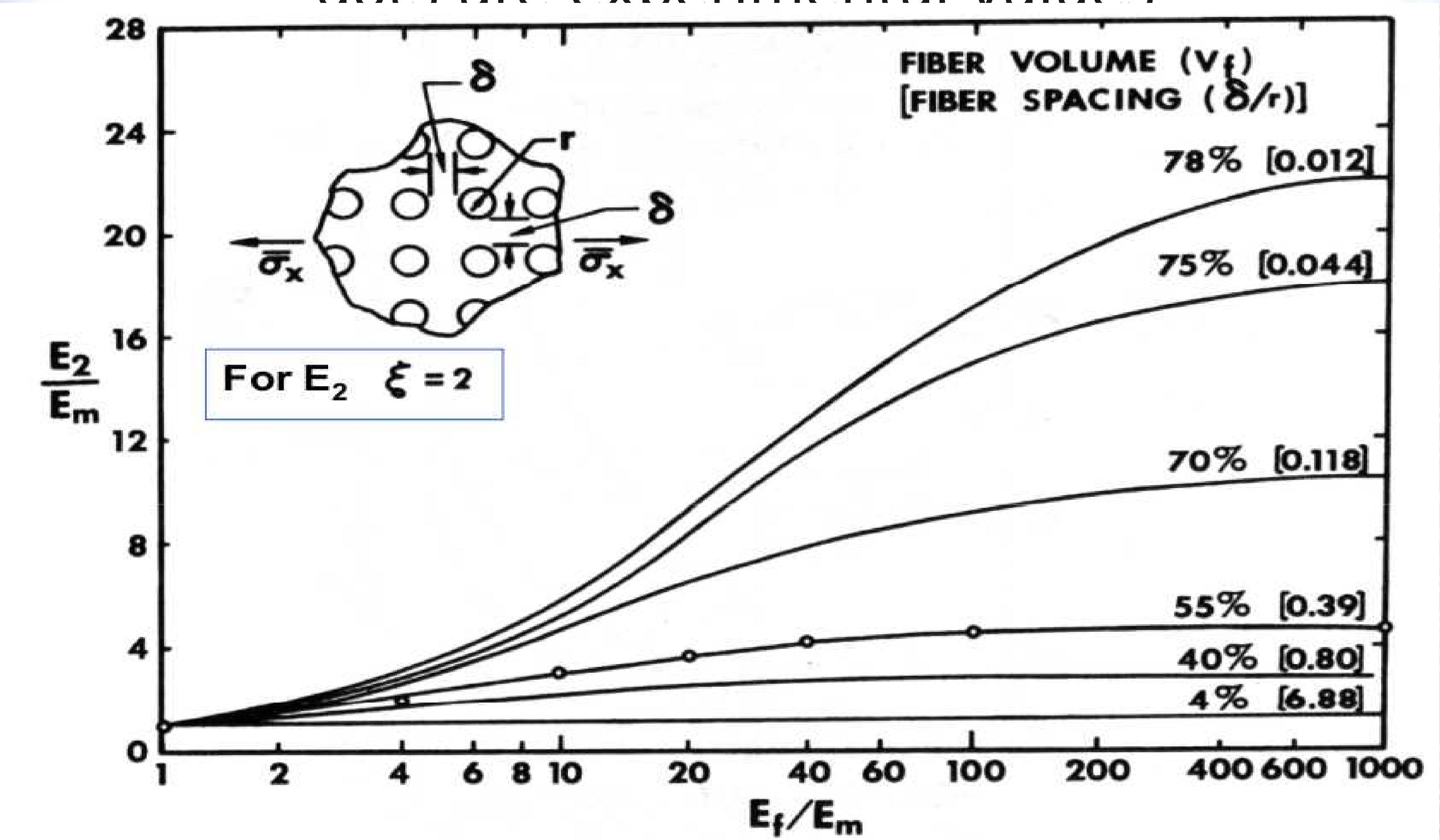
G_f and G_m approximated by

$$\frac{E}{2(1+\nu)}$$

Model	E_1 (GPa)	E_2 (GPa)	ν_{12}	G_{12} (GPa)
Voigt	41.8	41.8	0.22	17.0
Reuss	7.35	7.35	0.34	2.74
Hybrid VR	41.7	8.21	0.28	2.74
Sq. Fiber	41.7	10.8	0.26	3.64
Halpin-Tsai (round, sq. array)	41.7	13.1	0.28	3.93
Test Data*	39	8.6	0.28	3.8

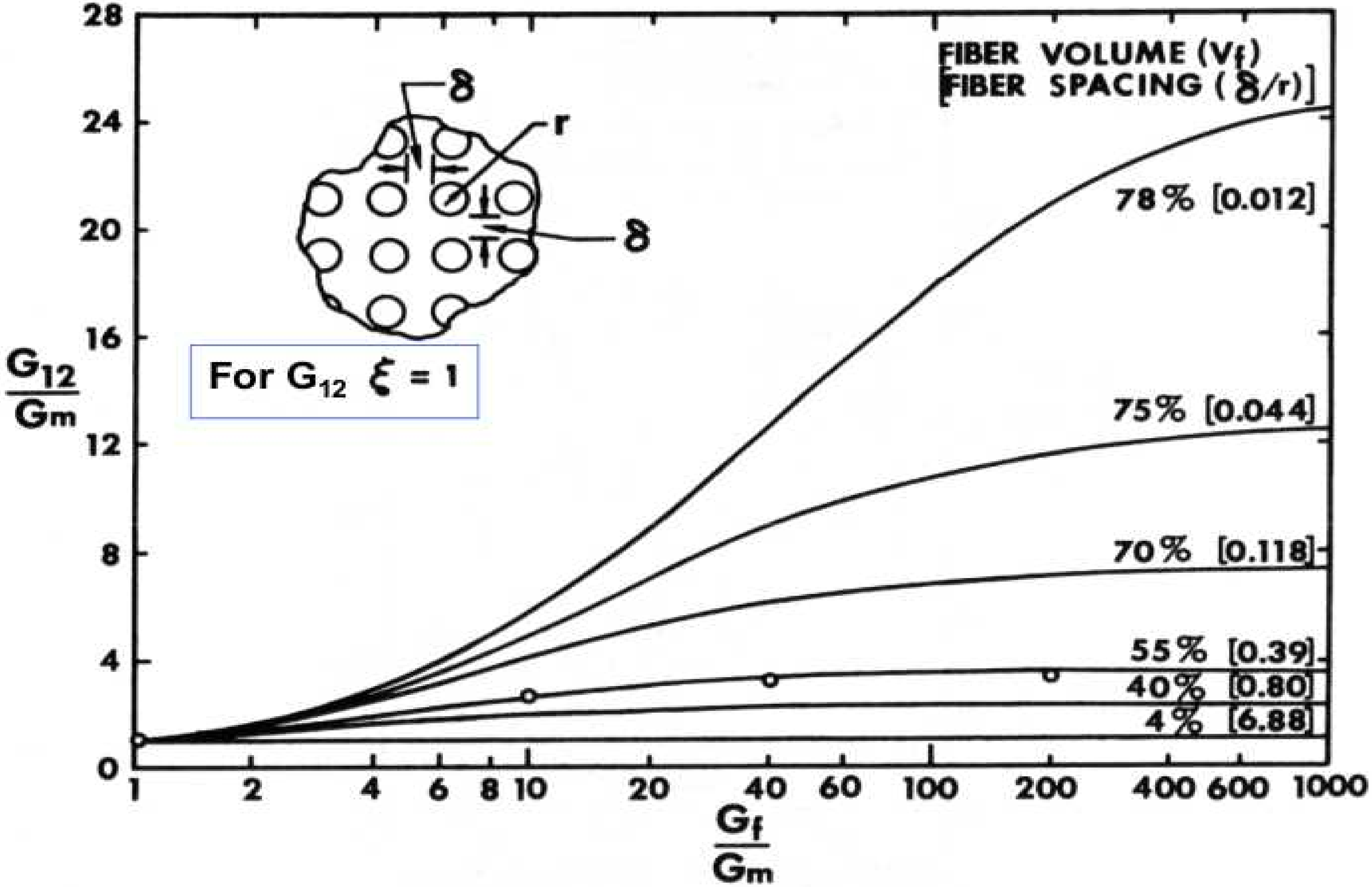
Transversal Young's modulus of corcular fibers

dots are experimental values

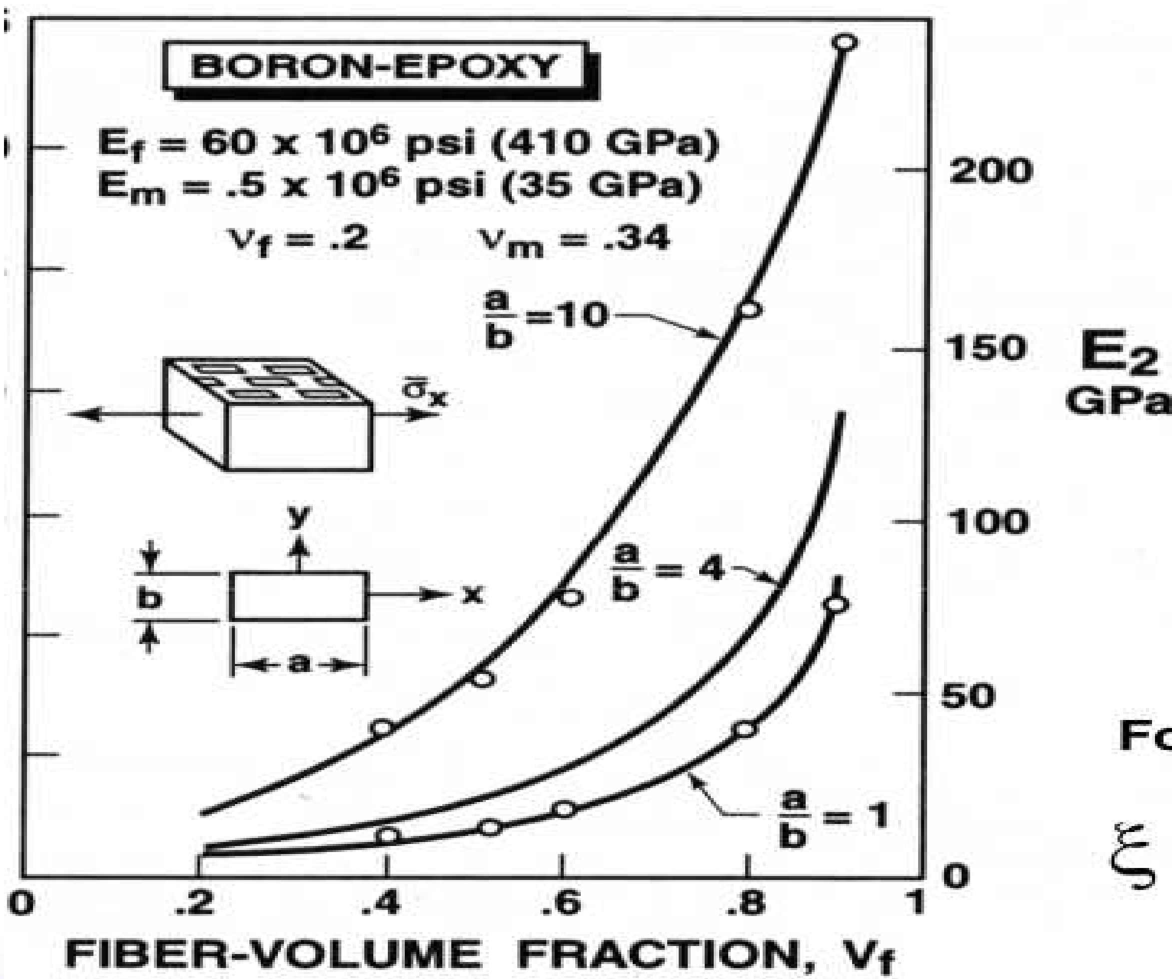


Shear modulus

in plain with main direction



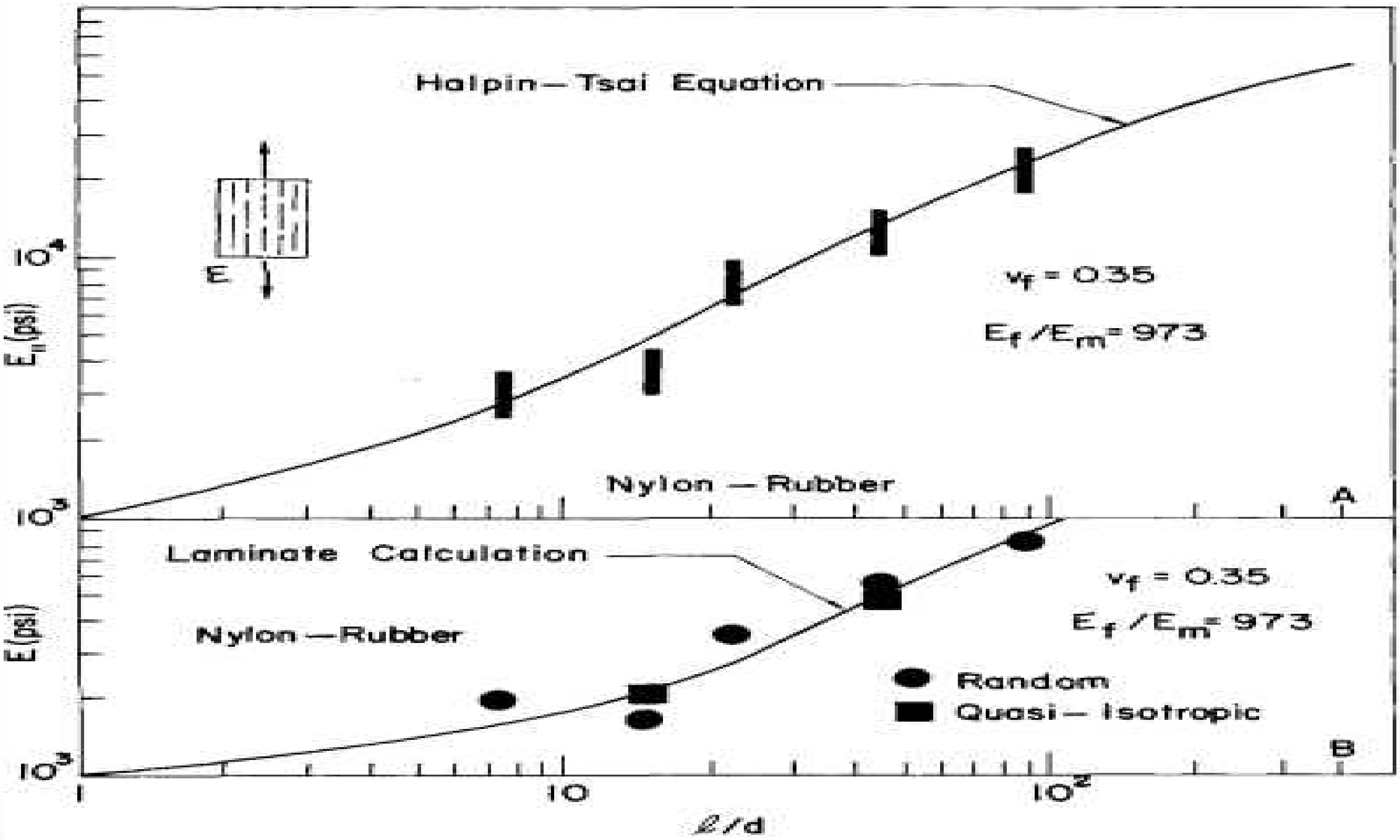
Transversal Young's modulus of rectangular fibers



For E_2

$$\xi = 2\frac{a}{b}$$

Longitunal Young's modulus by short fibers



Modified H-T model

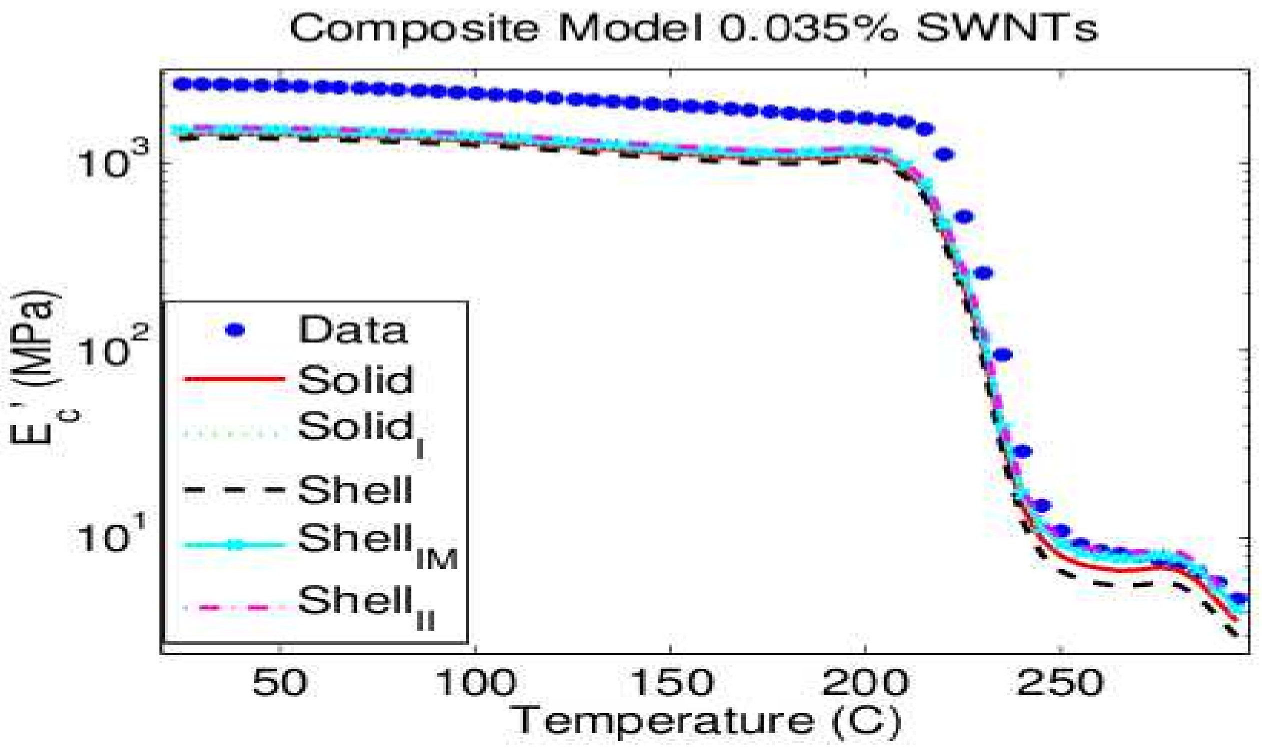
There was created for C nanotubes in polymer (polyimid). Nanotubes are randomly in plane (2D structure).

True relationship
$E_k = 3/8*E_l + 3/8*E_p + 2/8*G_{12}$

E_k, E_l a E_p are Young's modulus of 2D composite in plane of isotropy and of 1D composite longitudal and transversal.
G_{12} is shear modulus of 1D composite.

For nanotubes 0,8 nm diameter and length 1000 nm.

Comparison with experiment
(little of nanotubes, dots - experiment)



Comparison with experiment
(many of nanotubes, dots - experiment)